

# Stokes Equations

## Project 2

This project is closely related to Chapter 12 in *The finite element method: theory, implementation, and applications* by Larson and Bengzon (MGL). Read Chapter 12 carefully and take advantage of the analysis and implementation done there.

The motion of a highly viscous incompressible fluid enclosed within a domain  $\Omega \subset \mathbf{R}^2$  with boundary  $\partial\Omega$  and outward unit normal  $\mathbf{n}$  is governed by the Stokes equations: find the velocity  $\mathbf{u} : \Omega \rightarrow \mathbf{R}^2$  and the pressure  $p : \Omega \rightarrow \mathbf{R}$  such that

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \tag{1b}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \partial\Omega \tag{1c}$$

where  $\mathbf{f}$  is a given volume force. We assume that  $p$  satisfies the normalization constraint

$$(p, 1) = 0 \tag{2}$$

**Problem 1.** Derive the weak form of (1) by introducing the spaces

$$\mathbf{V} = \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v}|_{\partial\Omega} = \mathbf{0}\} \tag{3}$$

$$Q = \{q \in L^2(\Omega) : (q, 1) = 0\} \tag{4}$$

for the velocity and the pressure, respectively. Functions in  $Q$  are forced to have zero average. This is because  $(\nabla \cdot \mathbf{u}, q) = 0$  for constants  $q$ . Prove this statement.

**Problem 2.** Derive the finite element approximation by introducing abstract discrete spaces, using basis functions  $\{\varphi_i\}_{i=1}^n$  for the vector valued velocity component and  $\{\psi_i\}_{i=1}^m$  for the scalar valued pressure component.

**Problem 3.** Now consider the lid driven cavity problem. Let  $\Omega = [0, 1] \times [0, 1]$ ,  $u_1 = 1$  on the top boundary segment and  $u_1 = 0$  on all the other three, let  $u_2 = 0$  on all boundary segments, . and let  $f = 0$ . Implement the finite element method using the MINI element which is continuous piecewise linear basis functions for both velocity and pressure but with the pressure being enriched with the cubic polynomial (bubble function),

$$\varphi_4 = \varphi_1 \varphi_2 \varphi_3 \quad (5)$$

where  $\varphi_i$ ,  $i = 1, 2, 3$ , are the hat functions on an element  $K$ . Note that the condition  $(p, 1) = 0$  need to be enforced in the linear system using e.g. a Lagrangian multiplier  $\lambda$ , see page 307 in MGL. You may use the following subroutines.

```

clear all, close all
geom = square(0,1,0,1);
[p,e,t]=initmesh(geom,'hmax',0.05);
np=size(p,2); nt=size(t,2);
z=zeros(np+nt,1);
[A,Bx,By,a] = assemble(p,e,t);
A_tot=[1*A 0*A Bx z;
       0*A 1*A By z;
       Bx' By' sparse(np,np) a;
       z' z' a' 0];
b_tot = zeros(size(A_tot,1),1);
[A_tot,b_tot]=setbc(p,e,t,A_tot,b_tot);
x_tot=A_tot\b_tot;
xi=x_tot(1:np); eta=x_tot(np+nt+1:2*np+nt);
figure(1), pdeplot(p,e,t,'flowdata',[xi eta])
theta=x_tot(2*(np+nt)+1:end-1);
figure(2), pdesurf(p,t,theta)

function [A,Bx,By,a] = assemble(p,e,t)

```

```

np=size(p,2);  nt=size(t,2);
A=sparse(np+nt,np+nt);
Bx=sparse(np+nt,np);
By=sparse(np+nt,np);
a=zeros(np,1);
for i=1:nt
    % nodes, node coordinates, triangle area
    nodes=t(1:3,i);
    x=p(1,nodes);  y=p(2,nodes);
    dx=polyarea(x,y);
    % velocity degrees of freedom
    dofs=[nodes; np+i];
    % hat function gradients
    b=[y(2)-y(3); y(3)-y(1); y(1)-y(2)]/2/dx;
    c=[x(3)-x(2); x(1)-x(3); x(2)-x(1)]/2/dx;
    % element stiffness matrix
    AK=zeros(4,4);
    AK(1:3,1:3)=(b*b'+c*c')*dx;
    AK(4,4)=sum(sum((b*b'+c*c').*[2 1 1; 1 2 1; 1 1 2]*dx/180));
    A(dofs,dofs)=A(dofs,dofs)+AK;
    % element divergence matrix
    BK=zeros(4,3);
    BK(1:3,1:3)=-b*ones(1,3)*dx/3;
    BK(4,:)=(-[1 2 2; 2 1 2; 2 2 1]/60*dx*b)';
    Bx(dofs,nodes)=Bx(dofs,nodes)+BK;
    BK(1:3,1:3)=-c*ones(1,3)*dx/3;
    BK(4,:)=(-[1 2 2; 2 1 2; 2 2 1]/60*dx*c)';
    By(dofs,nodes)=By(dofs,nodes)+BK;
    % constraints to enforce mean value zero for the pressure
    a(nodes)=a(nodes)+ones(3,1)*dx/3;
end

function [A,b]=setbc(p,e,t,A,b)
np=size(p,2);  nt=size(t,2);
for i=1:np
    x=p(1,i);  y=p(2,i);
    if (x<0.001 | x>0.999 | y<0.001 | y>0.999) % cavity wall

```

```

A(i,:)=0;
A(i,i)=1; b(i)=0; % ux=0
A(np+nt+i,:)=0;
A(np+nt+i,np+nt+i)=1; b(np+nt+i)=0; % uy=0
end
if (y>0.999) % the lid of the cavity
    b(i)=1; % ux=1
end
end

function geom = square(xmin,xmax,ymin,ymax)
geom = [2 xmin xmax ymin ymax 1 0;
        2 xmax xmax ymin ymax 1 0;
        2 xmax xmin ymax ymax 1 0;
        2 xmin xmin ymax ymin 1 0]';

```

**Problem 4.** Verify that the null space for the matrix

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

consists of a single constant vector. This shows that there are no spurious pressure modes except the hydrostatic mode  $p_h = c$ ,  $c$  a constant, for this type of finite element and that the inf-sup condition is satisfied. *Hint:* See the help for the routine `null` and `full`. A quick way of obtaining  $B$  is to type `B=A_tot(1:2*(np+nt),2*(np+nt)+1:end-1)`.

**Problem 5.** Perform a convergence study by decreasing the mesh size for the  $L^2$  error in both velocity and pressure. What convergence rate do you get?

**Problem 6.** Simulate the double driven cavity with boundary conditions  $\mathbf{u} = 0$  on the lines  $x = 0$  and  $x = 1$ , and  $\mathbf{u} = [1, 0]$  on  $y = 0$  and  $y = 1$ .

**Problem 7.** Solve the lid and double driven cavity problems using COMSOL Multiphysics. Compare the results.

**Problem 8.** Feel free to further investigate your own code and/or COMSOL Multiphysics doing more experiments and investigations.