The time harmonic electric wave equation

Project 3

This project is closely related to Chapter 13 in *The finite element method: theory, implementation, and applications* by Larson and Bengzon (MGL). Read Chapter 13 carefully and take advantage of the analysis and implementation done there.

We consider the time harmonic electric wave equation for a scattered field E^s with $\mu = 1$,

$$\nabla \times \nabla \times \hat{E}^s + \kappa^2 \hat{E}^s = 0, \qquad \text{in } \Omega, \tag{1a}$$

$$\hat{E}^s \times n = g, \qquad \text{on } \partial\Omega. \tag{1b}$$

Problem 1. Derive equation (1) starting from the electric wave equation, with $\epsilon = 1$,

$$\nabla \times \nabla \times E + \ddot{E} = 0, \tag{2}$$

and the assumption that $E = \hat{E}e^{i\omega t}$ and $\hat{E} = \hat{E}_s + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\omega x_2}$.

Problem 2. Derive the weak form of (1) by introducing the spaces

$$\boldsymbol{V}_0 = \{\boldsymbol{v} \in [L^2(\Omega)]^2 : \nabla \times v \in [L^2(\Omega)]^2 : n \times v|_{\partial\Omega} = 0\}. \tag{3}$$

Note that $\hat{E}^s \notin V_0$ but given any function $v_g \in H(\text{curl}; \Omega)$ equal to g on the boundary there is a $\hat{E}^s_0 \in V_0$ such that $\hat{E}^s = \hat{E}^s_0 + v_g$.

Problem 3. Derive the finite element approximation by the Nédélec finite element space with basis functions $\{S_j^{\text{ND}}\}_{j=1}^{n_e}$.

Problem 4. Now consider the scattering example presented in Section 13.5.5.1. Implement the solver and reproduce the pictures in the book.

Problem 5. Perform a convergence study by decreasing the mesh size for the L^2 error in $\hat{E}^s$. What convergence rate do you get?

Problem 6. Study how the solution depends on the size of the perfectly matched layer. Increase the size and study the convergence.

Problem 7. Set up the same problem using COMSOL Multiphysics. Compare the result.

Problem 8. Set up more complicated geometries and study the scattering pattern using COMSOL Multiphysics.

Problem 9. Feel free to further investigate your own code and/or COMSOL Multiphysics doing more experiments and investigations.