Solutions to exercises from Chapters 7-10 in MGL.
Axel Målqvist 2013-01-08
(email `axel.malqvist@it.uu.se` if you find mistakes)

Exercise 7.1 Calculate $\sum_{|\alpha|=2} D^\alpha u$.
Sol. $\sum_{|\alpha|=2} D^\alpha u = \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u$.

Exercise 7.2 Calculate the weak derivative of $g(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$.
Sol. Using integration by parts and that $\varphi(0) = \varphi(2) = 0$ we get,

$$
-\int_0^2 g \cdot \varphi'(x)\, dx = -\int_0^1 x \cdot \varphi'(x)\, dx - \int_1^2 \varphi'(x)\, dx
$$

$$
= \int_0^1 \varphi(x)\, dx - 1 \cdot \varphi(1) - \varphi(2) + \varphi(1)
$$

$$
= \int_0^2 D_w g(x) \varphi(x)\, dx,
$$

where the weak derivative $D_w g(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$.

Exercise 7.3 Write down the inner product and norm of $H^2(\Omega)$.
Sol. We have,

$$
(u, v)_{H^2(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i} u, \frac{\partial}{\partial x_i} v \right)_{L^2(\Omega)}
$$

$$
+ \sum_{i=1}^{d} \sum_{j=1}^{d} \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v \right)_{L^2(\Omega)},
$$

and $\|v\|_{H^2(\Omega)} = (v, v)_{H^2(\Omega)}^{1/2}$.

Exercise 7.4 Does $L^2(\Omega) \subset H_0^1(\Omega)$? Does $H^2(\Omega) \subset H_0^1(\Omega)$?
Sol. No. A counter example is a function $f(x) = \begin{cases} 1, & \text{on } \omega \subset \Omega \\ 0. & \text{on } \Omega \setminus \omega \end{cases}$ which is in $L^2(\Omega)$ but not in $H_0^1(\Omega)$. Also the second question has negative answer since functions in $H^2(\Omega)$ do not in general have zero trace.

Exercise 7.5 Let $v(x) = \log(\log(|x|^{-1}))$ on a disc Ω with radius e^{-1}. Verify that $v \in H^1(\Omega)$. Is $v \in C(\Omega)$?
Sol. We use polar coordinates to get,

$$\|\nabla v\|_{L^2(\Omega)}^2 = 2\pi \int_0^{1/e} \frac{r\,dr}{r^2(\log r)^2} = \{r = e^y\} = 2\pi \int_{-\infty}^{-1} y^{-2}\,dy = 2\pi.$$

Furthermore, $v|_{\partial\Omega} = 0$ i.e. $v \in H_0^1(\Omega) \subset H^1(\Omega)$. However, $v \notin C(\Omega)$ since it is unbounded on a compact domain.

Exercise 7.6 Consider $C(I)$ with the supremum norm, where $I = [0, 1]$. Let $f(x) = 1$ and $g(x) = x$. Does the parallelogram law hold for f and g? Is $C(I)$ an inner product space?
Sol. No since, $\|f+g\|_{L^\infty(I)} + \|f-g\|_{L^\infty(I)} = 3$ and $2\|f\|_{L^\infty(\Omega)}^2 + 2\|g\|_{L^\infty(\Omega)}^2 = 4$. It is not an inner product space.

Exercise 7.7 Show that $\|u\|_V \leq m^{-1}C_l$.
Sol. We have $\|u\|_V^2 \leq m^{-1}a(u, u) = m^{-1}l(u) \leq m^{-1}C_l\|u\|_V$.

Exercise 7.8 Show that $\|v\|_{H^1(\Omega)}$ and $|v|_{H_0^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}$ are equivalent norms on $H_0^1(\Omega)$.
Sol. We directly get $|v|_{H_0^1(\Omega)} \leq \|v\|_{H^1(\Omega)}$. Using the Poincare inequality we also have,

$$\|v\|_{H^1(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 \leq (C+1)\|\nabla v\|_{L^2(\Omega)}^2 = (C+1)|v|_{H_0^1(\Omega)}^2,$$

which proves the equivalence.

Exercise 7.9 Compute m, C_a, C_l for the problem $-\Delta u = xy^2$ on $\Omega = [-1, 2] \times [0, 3]$ with homogeneous Dirichlet boundary conditions.
Sol. We have $a(u, v) = (\nabla u, \nabla v)$ for all $u, v \in H_0^1(\Omega)$, i.e. $m = C_a = 1$ since $a(u, v) \leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$ and $\|\nabla v\|_{L^2(\Omega)}^2 \leq a(v, v)$ for all $u, v \in H_0^1(\Omega)$. Furthermore, use integration by parts w.r.t. x to get,

$$l(v) = \int_\Omega xy^2 v\,dxdy = -\int_\Omega y^2 v_x'\,dxdy \leq \|y^2\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

We note that $C_l = 27 \cdot 5^{-1/2}$ but possibly less.

2

Exercise 7.10. Let A be a real $n \times n$ matrix which is coercive with respect to the Euclidean norm. Further let b be a real n vector. What can be said about the eigenvalues of A? Show that $Au = b$ has a unique solution.
Sol.
Let (u, λ) be an eigenpair solving $Au = \lambda u$, normalized so that $u \cdot \bar{u} = 1$. Since A is real we note that $A\bar{u} = \bar{\lambda}\bar{u}$, i.e. complex conjugate eigenvalues and eigenvectors. In particular this means that $u + \bar{u}, i(u - \bar{u}) \in \mathbb{R}^n$. We now use that A is coercive on $\mathbb{R}^n$, i.e. $v \cdot Av \geq c v \cdot v$ for all $v \in \mathbb{R}^n$, in the following calculation,

$$
\begin{aligned}
4c = {}& 4c\, u \cdot \bar{u} \\
= {}& c(u + \bar{u}) \cdot (u + \bar{u}) - c(u - \bar{u}) \cdot (u - \bar{u}) \\
= {}& c(u + \bar{u}) \cdot (u + \bar{u}) + c\, i(u - \bar{u}) \cdot i(u - \bar{u}) \\
\leq {}& (u + \bar{u}) \cdot A(u + \bar{u}) + i(u - \bar{u}) \cdot Ai(u - \bar{u}) \\
= {}& (u + \bar{u}) \cdot A(u + \bar{u}) - (u - \bar{u}) \cdot A(u - \bar{u}) \\
= {}& 2u \cdot A\bar{u} + 2\bar{u} \cdot Au \\
= {}& 2(\lambda + \bar{\lambda})u \cdot \bar{u} \\
= {}& 2(\lambda + \bar{\lambda}).
\end{aligned}
$$

We conclude that $\Re(\lambda) \geq c$. Since zero is not an eigenvalue, A is invertible and there exists a unique solution to the linear system.

Exercise 7.11 Verify the trace inequality for $v = x$ on $\Omega = [0, L] \times [0, L]$. How does the constant depend on L?
Sol. We start with the left hand side $\|v\|_{L^2(\partial\Omega)} = 5^{1/2}3^{-1/2}L^{3/2}$. For the right hand side we have $\|v\|_{L^2(\Omega)}^{1/2} = 3^{-1/4}L$ and $\|\nabla v\|_{L^2(\Omega)}^{1/2} = L^{1/2}$. We conclude that there is a constant C, independent of L, for which the trace inequality is fulfilled.

Exercise 7.12 Consider $-\nabla \cdot (a\nabla u) = 1$ on the unit square with homogeneous Dirichlet boundary conditions. Let a be the 2×2 matrix with entries $[4, 1; 1, 2]$. Decide if the requirements for the Lax-Milgram Lemma is fulfilled.
Sol. The bilinear form is $b(v, w) = (a\nabla v, \nabla w)$. The eigenvalues of a are $1 < \lambda_1 < \lambda_2 < 5$. It is clear that $\lambda_1\|\nabla v\|_{L^2(\Omega)}^2 \leq b(v, v)$ for all $v \in H_0^1(\Omega)$ i.e. the bilinear form is coercive. It is also clear that $b(v, w) \leq \lambda_2\|\nabla v\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}$ i.e. the bilinear form is continuous. Let $l(v) = \int_\Omega 1 \cdot v\, dx \leq |\Omega|^{1/2}\|v\|_{L^2(\Omega)} \leq$

$C\|\nabla v\|_{L^2(\Omega)}$ for all $v \in H_0^1(\Omega)$, using the Poincare inequality. The assumptions of the Lax-Milgram Lemma are therefore fulfilled.

Exercise 8.1 Work out the formula for the cubic Lagrange shape functions on the reference triangle.

Sol. Let $\varphi_1 = 1 - r - s$, $\varphi_2 = r$, and $\varphi_3 = s$ be the linear shape functions using the coordinates of Figure 8.2. We get,

$$
\begin{aligned}
S_1 &= \varphi_1(3\varphi_1 - 1)(3\varphi_1 - 2)/2, \\
S_2 &= \varphi_2(3\varphi_2 - 1)(3\varphi_2 - 2)/2, \\
S_3 &= \varphi_3(3\varphi_3 - 1)(3\varphi_3 - 2)/2, \\
S_4 &= 9\varphi_2\varphi_3(3\varphi_2 - 1)/2, \\
S_5 &= 9\varphi_2\varphi_3(3\varphi_3 - 1)/2, \\
S_6 &= 9\varphi_3\varphi_1(3\varphi_3 - 1)/2, \\
S_7 &= 9\varphi_3\varphi_1(3\varphi_1 - 1)/2, \\
S_8 &= 9\varphi_1\varphi_2(3\varphi_1 - 1)/2, \\
S_9 &= 9\varphi_1\varphi_2(3\varphi_2 - 1)/2, \\
S_{10} &= 27\varphi_1\varphi_2\varphi_3.
\end{aligned}
$$

Exercise 8.2 Calculate the entries of the 4×4 element stiffness matrix using bilinear shape functions on $K = [0, h] \times [0, h]$.

Sol. Let $h = 1$ and $\hat{K} = [0, 1] \times [0, 1]$. We have four basis functions $\varphi_1 = (1 - x)(1 - y)$, $\varphi_2 = x(1 - y)$, $\varphi_3 = xy$, and $\varphi_4 = (1 - x)y$. The gradients of the basis functions are $\nabla\varphi_1 = \begin{bmatrix} y - 1 \\ x - 1 \end{bmatrix}$, $\nabla\varphi_2 = \begin{bmatrix} 1 - y \\ -x \end{bmatrix}$, $\nabla\varphi_3 = \begin{bmatrix} y \\ x \end{bmatrix}$, and $\nabla\varphi_4 = \begin{bmatrix} -y \\ 1 - x \end{bmatrix}$. Due to symmetry we only need to compute three distinct integrals namely $\int_{\hat{K}} \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx = 2/3$, $\int_{\hat{K}} \nabla\varphi_1 \cdot \nabla\varphi_2 \, dx = \int_{\hat{K}} \nabla\varphi_1 \cdot \nabla\varphi_4 \, dx = -1/6$, and $\int_{\hat{K}} \nabla\varphi_1 \cdot \nabla\varphi_3 \, dx = -1/3$. We get,

$$
\frac{1}{6}
\begin{bmatrix}
4 & -1 & -2 & -1 \\
-1 & 4 & -1 & -2 \\
-2 & -1 & 4 & -1 \\
-1 & -2 & -1 & 4
\end{bmatrix}.
$$

4

Scaling with h does not effect the element stiffness matrix since the area and the scaling of the gradients match exactly.

Exercise 8.3 Show that the bilinear element is not unisolvent if the four nodes are placed on $x_1 = (-1, 0)$, $x_2 = (0, -1)$, $x_3 = (1, 0)$, and $x_4 = (0, 1)$ on the reference square $[-1, 1] \times [-1, 1]$.
Sol. The polynomial function space $P = \text{span}(\{1, x, y, xy\})$. We want to show that there is an $v \neq 0$ such that $v(x_i) = 0$ for $i = 1, \ldots, 4$. We immediately see that $v = xy$ fulfills this requirement.

Exercise 8.5 Draw the shape functions for the Crouzeix-Raviart element on the reference triangle.
Sol. The shape functions on the reference triangle with corners $(0, 0)$, $(0, 1)$, and $(1, 0)$ are $S_1 = 2x + 2y - 1$, $S_2 = 1 - 2x$, and $S_3 = 1 - 2y$.

Exercise 8.6 Calculate the Crouzeix-Raviart interpolant of $f = 2xy + 4$ on the reference triangle.
Sol. Let the basis functions be $\psi_1 = 2x + 2y - 1$, $\psi_2 = 1 - 2x$, and $\psi_3 = 1 - 2y$. Note that these functions are one in one edge midpoint and zero in the other two. We get $\pi_{CR}f = f(1/2, 1/2)\psi_1 + f(0, 1/2)\psi_2 + f(1/2, 0)\psi_3 = 9x + 9y - 9/2 + 4 - 8x + 4 - 8y = x + y + 7/2$.

Exercise 8.7 Show that $\nabla \times S_i^{\text{ND}} = 2|E_i|\nabla\varphi_j \times \nabla\varphi_k$.
Sol. We note that $\nabla \times \nabla v = 0$ for all $v \in C^2$ and use the product rule to get, $\nabla \times (|E_i|(\varphi_j\nabla\varphi_k - \varphi_k\nabla\varphi_j)) = |E_i|(\nabla\varphi_j \times \nabla\varphi_k - \nabla\varphi_k \times \nabla\varphi_j) = 2|E_i|\nabla\varphi_j \times \nabla\varphi_k$.

Exercise 8.8 How does the iso-parametric map look in three dimensions?
Sol. Let $\bar{K}$ be the reference tetrahedron with corners in $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Let K be an element defined by the nodes $N_i = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$, $i = 1, \ldots, n$ and the associated shape functions S_j through

the following map:

$$x_1(r, s, t) = \sum_{i=1}^{n} x_1^{(i)} S_i(r, s, t),$$

$$x_2(r, s, t) = \sum_{i=1}^{n} x_2^{(i)} S_i(r, s, t),$$

$$x_3(r, s, t) = \sum_{i=1}^{n} x_3^{(i)} S_i(r, s, t),$$

for all $r, s, t \in \bar{K}$.

Exercise 9.1 Show that Newton's method converges in a single iteration for a linear problem $Ax = b$.

Sol. Let $g(x) = Ax - b$. We note that the Jacobian $Dg = A$. We get $x^1 = x^0 - A^{-1}(Ax^0 - b) = A^{-1}b = x$, independent of x^0.

Exercise 9.2 Derive Newton's method for the following non-linear problems: $-\Delta u = u - u^3$, $-\Delta u + sin(u) = 1$, $-\nabla \cdot ((1 + u^2)\nabla u) = 1$, and $-\Delta u = f(u)$, where $f(u)$ is a differentiable function.

Sol. Let V be an appropriate function space for the problem.

(a) Let $u^{k+1} = u^k + \delta u$, where $\delta u \in V$ is given as the solution to,

$$(\nabla \delta u, \nabla v) + ((3(u^k)^2 - 1)\delta u, v) = (u^k - (u^k)^3, v) - (\nabla u^k, \nabla v), \quad \forall v \in V.$$

(b) Let $u^{k+1} = u^k + \delta u$, where $\delta u \in V$ is given as the solution to,

$$(\nabla \delta u, \nabla v) + (\cos(u^k)\delta u, v) = -(\sin(u^k), v) - (\nabla u^k, \nabla v), \quad \forall v \in V.$$

(c) Let $u^{k+1} = u^k + \delta u$, where $\delta u \in V$ is given as the solution to,

$$((1 + (u^k)^2)\nabla \delta u, \nabla v) + (2u^k \nabla u^k \delta u, \nabla v) = -((1 + (u^k)^2)\nabla u^k, \nabla v), \quad \forall v \in V.$$

(d) Let $u^{k+1} = u^k + \delta u$, where $\delta u \in V$ is given as the solution to,

$$(\nabla \delta u, \nabla v) - (Df(u^k)\delta u, v) = (f(u^k), v) - (\nabla u^k, \nabla v), \quad \forall v \in V.$$

Exercise 9.4 Derive Newton's method for the Predator-prey system.

Sol. Let $u_i^{k+1} = u_i^k + \delta u_i$, $i = 1, 2$, where $\delta u_1, \delta u_2 \in V_0 = \{v \in V : v|_{\partial\Omega} = 0\}$ solves,

$$(\nabla \delta u_1, \nabla v) - ((1 - u_2^k)\delta u_1, v) + (u_1^k \delta u_2, v) = -(\nabla u_1^k, \nabla v) + (u_1^k(1 - u_2^k), v),$$
$$(\nabla \delta u_2, \nabla v) - (u_2^k \delta u_1, v) + ((1 - u_1^k)\delta u_2, v) = -(\nabla u_2^k, \nabla v) - (u_2^k(1 - u_1^k), v),$$

for all $v \in V_0$.

Exercise 10.1 Compute the least squares solution to a linear system $Ax = b$. Which norm is minimized by this solution?

Sol. We solve the normal equations $x = (A^T A)^{-1} A^T b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The residual $Ax - b$ is minimized in the Euclidean norm.

Exercise 10.2 Verify that the standard FEM for $-\epsilon u_x x + u_x = 1$ on $0 < x < 1$ and $u(0) = u(1) = 0$ is,

$$-\epsilon \frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{h^2} + \frac{\xi_{i+1} - \xi_{i-1}}{2h} = 1. \tag{1}$$

where $i = 1, 2, \ldots, n - 1$.

Sol. We fix the text function to φ_i and note that only three functions in the trail space overlap, namely φ_{i+1}, φ_i, and φ_{i-1}. The finite element method now gives us the following relation,

$$\sum_{j=i-1}^{i+1} \xi_j \left(\epsilon(\varphi_j', \varphi_i') + (\varphi_j', \varphi_i) \right) = (1, \varphi_i), \quad i = 1, \ldots, n - 1.$$

We compute the integrals $(\varphi_{i+1}', \varphi_i') = -h^{-1}$, $(\varphi_i', \varphi_i') = 2h^{-1}$, $(\varphi_{i-1}', \varphi_i') = -h^{-1}$, $(\varphi_{i+1}', \varphi_i) = 0.5$, $(\varphi_i', \varphi_i) = 0$, $(\varphi_{i-1}', \varphi_i') = -0.5$, and $(1, \varphi_i) = h$ for $i = 1, \ldots, n - 1$. Equation (1) follows immediately.

Exercise 10.3 Derive the GLS method for,

$$-\epsilon \Delta u + b \cdot \nabla u + cu = f, \quad x \in \Omega \quad u = 0, \quad x \in \partial\Omega.$$

Describe the arising linear system.

Sol. The GLS approximation is given as solution to, find $u_h \in V_h$ such that,

$$(\epsilon \nabla u_h, \nabla v) + (b \cdot \nabla u_h, v) + (cu_h, v) + \delta \sum_{K \in \mathcal{K}} (b \cdot \nabla u_h + cu_h, b \cdot \nabla v + cv)$$
$$= (f, v) + \delta(f, b \cdot \nabla v + cv),$$

for all $v \in V_h$, where $\delta = \begin{cases} Ch^2, & \text{if } \epsilon > h \\ Ch\|b\|_{L^\infty}^{-1}(\Omega), & \text{if } \epsilon \leq h \end{cases}$. The matrix is non-symmetric and has an additional symmetric positive term.

Exercise 10.6 Show that $\||v\||^2 = \epsilon \|\nabla v\|_{L^2(\Omega)}^2 + \delta \|b \cdot \nabla v\|_{L^2(\Omega)}^2$ is a norm on $H_0^1(\Omega)$.

Sol. (i) We have

$$\||\lambda v\||^2 = \epsilon \|\nabla \lambda v\|_{L^2(\Omega)}^2 + \delta \|b \cdot \nabla \lambda v\|_{L^2(\Omega)}^2$$
$$= |\lambda|^2 \left(\epsilon \|\nabla v\|_{L^2(\Omega)}^2 + \delta \|b \cdot \nabla v\|_{L^2(\Omega)}^2 \right) = |\lambda|^2 \||v\||^2.$$

(ii) We let $\langle v, w \rangle = \epsilon(\nabla v, \nabla w) + \delta(b \cdot \nabla v, b \cdot \nabla w)$ and note that,

$$\langle v, w \rangle \leq \epsilon \|\nabla v\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} + \delta \|b \cdot \nabla v\|_{L^2(\Omega)} \|b \cdot \nabla w\|_{L^2(\Omega)}$$
$$\leq (\epsilon \|\nabla v\|_{L^2(\Omega)}^2 + \delta \|\nabla v\|_{L^2(\Omega)}^2)^{1/2} \cdot (\epsilon \|\nabla w\|_{L^2(\Omega)}^2 + \delta \|\nabla w\|_{L^2(\Omega)}^2)^{1/2}$$
$$= \||v\|| \cdot \||w\||,$$

since $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$ for $a, b, c, d \in \mathbf{R}$. We conclude.

$$\||v + w\||^2 = \langle v, v \rangle + \langle w, w \rangle + 2\langle v, w \rangle \leq (\||v\|| + \||w\||)^2.$$

(iii) It is clear that $\||v\|| \geq 0$. Furthermore $\||0\|| = 0$. Finally if $\||v\|| = 0$ we have $\|\nabla v\|_{L^2(\Omega)} = 0$ so v is constant but the only constant in $H_0^1(\Omega)$ is zero so $v = 0$.

Exercise 10.7 Show that $a_h(v, w) = \epsilon(\nabla v, \nabla w) + (b \cdot \nabla v, w) + \delta(b \cdot \nabla v, b \cdot \nabla w)$ and $l_h(v) = (f, v) + \delta(f, b \cdot \nabla v)$ are continuous on V_h.

Sol. We have $a_h(v, w) = \langle v, w \rangle + (b \cdot \nabla v, w) \leq \||v\|| \cdot \||w\|| + \|b \cdot \nabla v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \leq (1 + C\delta^{-1/2}\epsilon^{-1/2})\||v\|| \cdot \||w\||$. Furthermore, $l_h(v) \leq \|f\|_{L^2(\Omega)}(\|v\|_{L^2(\Omega)} + \delta \|b \cdot \nabla v\|_{L^2(\Omega)}) \leq (\epsilon^{-1/2} + \delta^{1/2})\|f\|_{L^2(\Omega)} \||v\||$.