UPPSALA UNIVERSITET

Institionen för informationsteknologi Richard Wait

Besöksadress:
MIC, Polacksbacken
Lägerhyddvägen 2

Postadress:
Box 337
751 05 Uppsala

Telefon:
018–471 2757

Telefax:
018–52 30 59

Hemsida:
http://user.it.uu.se/~richard

Epost:
richard@it.uu.se

Department of Information Technology Richard Wait

Visiting address:
MIC, Polacksbacken
Lägerhyddvägen 2

Postal address:
Box 337
SE-751 05 Uppsala
SWEDEN

Telephone:
+46 18–471 2757

Telefax:
+46 18–52 30 59

Web page:
http://user.it.uu.se/~richard

Email:
richard@it.uu.se

Poisson Processes and Differential Equations

A Poisson Process is a continuous time Markov Chain $\{X(t)\}$ with discrete states

$$X(t) \in \{0, 1, 2, 3, \ldots\} \quad t \in [0, \infty)$$

Assume that in a population,

- the probability that an individual gives birth in time interval Δt is $b\Delta t$
- the probability of more than one birth is negligible ($\mathcal{O}(\Delta t^2)$ terms are ignored) (Note this is another use of the big Oh! notation.)

So assuming there are no deaths, in a population of size n, define

$$G(t) = prob\{\text{There is no change in the population up to time } t\}$$

Then

$$prob\{\text{There is no change in the population from time } t \text{ to time } t + \Delta t\} = 1 - bn\Delta t$$

and so

$$G(t + \Delta t) = G(t)(1 - bn\Delta t)$$

$$G(t + \Delta t) - G(t) = -G(t)bn\Delta t$$

$$\frac{G(t + \Delta t) - G(t)}{\Delta t} = -G(t)bn$$

so in the limit as $\Delta t \to 0$ this becomes the *differential equation*

$$\frac{dG(t)}{dt} = -bnG(t) \quad t > 0$$

with the *initial condition* $G(0) = 1$. This is a well posed problem

Differential equation + Initial Condition = Initial Value Problem (IVP)

The solution is

$$G(t) = e^{-bnt}$$

Given $\int_0^\infty G(t) = \frac{1}{bn}$, then $f(t) = bnG(t)$ is a probability density function. For a random variable t defined by this p.d.f, $E(t)$ is known as the *mean inter-arrival time*

$$E(t) = \int_0^\infty tf(t)dt = \frac{1}{bn}$$

This shows that on average the population grows from size n to size $n+1$ in time $\Delta t = \dfrac{1}{bn}$ or the population size $n(t)$ at time t satisfies

$$n(t+\Delta t) - n(t) = 1 \quad \Rightarrow \quad \frac{n(t+\Delta t) - n(t)}{\Delta t} = \frac{1}{\Delta t}$$

again this can be replaced by a differential equation

$$\frac{dn(t)}{dt} = bn \quad t > 0$$

Given the initial population $n(0)$ the solution of the IVP can be written as

$$n(t) = e^{bt} n(0) \quad t \geq 0$$

Random Numbers

The *MATLAB* function `rand` generates numbers that are distributed uniformly on the unit interval, *i.e.*

$$p(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

In the Poisson process above, $p(t) = bne^{-bnt} \quad t \geq 0$. In a uniform distribution on $[0,1]$, $F(x) = x$ for Poisson, $F(t) = 1 - e^{-bnt}$. If the number generated by `rand` is $1 - x$ then a transformation from x to t is

$$x = e^{-bnt}$$

or

$$t = -\frac{\log(x)}{bn}$$

So in a *simulation* of the population growth if $T(i)$ is the time of the i-th birth, assuming the initial population size is $n(0) = N$ then at $T(i+1)$ the population grows from $N+i$ to $N+i+1$ where

$$T(i+1) = T(i) - \frac{log(x)}{b(N+i)} \quad i = 0, 1, \ldots$$

Assignment

Write a *MATLAB* program that plots on the same graph, the solution of the IVP and 5 random simulations.

- Construct the graphs with $b = 1, N = 1$ for a population up to size 50.
- Construct the graphs with $b = .01, N = 1000$ for a population up to size 10000.

Plot the random simulations using `stairs`. Your assignment report should include a copy of the *MATLAB* code and the graphs. Assignments to be submitted in the *FUSSMOBB IN* box at the latest Thursday 5th October.