

8 Räkneövningar till kapitel 8

8.1

The problem of collinearity.
Consider the following model

$$\hat{y}(k) = au_1(k) + bu_2(k)$$

Here $u_1(k)$ and $u_2(k)$ are two measured input signals. Suppose that the data is generated by

$$y(k) = a_0u_1 + bu_2$$

where $u_1 = K$ and $u_2 = L$, that is two constant signals with amplitude K and L are used as input signals. Show that $\det \Phi^T \Phi = 0$, and hence the least squares method can not be used.

Remark:

The columns in Φ must be linearly independent for $(\Phi^T \Phi)^{-1}$ to exist.

8.2

Calculating the least squares estimate for ARX models.
Consider the following ARX model:

$$y(k) + ay(k-1) = bu(k-1) + e(k) \quad (1)$$

($e(k)$ is white noise with zero mean)

Assume that available data are : $y(1), u(1), y(2), u(2), \dots, y(102), u(102)$ and the following sums have been calculated:

$$\sum_{k=2}^{102} y^2(k-1) = 5.0, \quad \sum_{k=2}^{102} y(k-1)u(k-1) = 1.0, \quad \sum_{k=2}^{102} u^2(k-1) = 1.0,$$

$$\sum_{k=2}^{102} y(k)y(k-1) = 4.5, \quad \sum_{k=2}^{102} y(k)u(k-1) = 1.0,$$

Which value of $\theta = (a \ b)^T$ minimizes the quadratic criteria

$$V_N(\theta) = \frac{1}{N} \sum_{k=2}^N (y(k) - \hat{y}(k, \theta))^2$$

where $\hat{y}(k, \theta)$ is the predictor obtained from the ARX model (1)?

8.3

Optimal input signal.

The following system is given:

$$y(k) = b_0 u(k) + b_1 u(k-1) + e(k)$$

$e(k)$ is white noise with zero mean and variance λ .

The parameters are estimated with the least squares method. Consider the case when the number of data points N goes to infinity (in practice, this means that we have many data points available)

a) Show that $\text{var}(\hat{b}_0)$ and $\text{var}(\hat{b}_1)$ only depends on the following values of the covariance function:

$$R_u(0) = Eu^2(k) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N u^2(k)$$

$$R_u(1) = Eu(k)u(k-1) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N u(k)u(k-1)$$

b) Assume that the energy of the input signal is constrained to

$$R_u(0) = Eu^2(k) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N u^2(k) \leq 1$$

Determine $R_u(0)$ and $R_u(1)$ so that the variance of the parameter estimates are minimized.

8.4

The following system is given:

$$y(k) = b_1 u(k-1) + b_2 u(k-2) + e(k)$$

$e(k)$ is white noise with zero mean and variance λ .

Assume that the number of data points goes to infinity.

a) Assume that $u(k)$ is white noise¹ with variance σ and zero mean. Show that the least squares estimate converge to the true system parameters.

b) Assume that $u(k)$ is a unit step: $u(k) = 0, k \leq 0$, and $u(k) = 1, k \geq 1$, show that the matrix $\bar{R} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N \varphi(k) \varphi^T(k)$ becomes singular.

¹In general, we will also assume that $e(k)$ and $u(k)$ are uncorrelated if not explicitly stated otherwise.

8.5

The following system is given:

$$y(k) = b_1 u(k-1) + b_2 u(k-2) + e(k)$$

$e(k)$ is white noise with zero mean and variance λ .

The following predictor

$$\hat{y}(k) = bu(k-1)$$

is used to estimate the parameter b with the least squares (LS) method.

Calculate the LS estimate of b (expressed in b_1 and b_2) as the number of data points goes to infinity for the cases:

- a) The input signal $u(k)$ is white noise.
- b) The input signal is a sinusoidal $u(k) = A \sin(\omega_1 t)$ which has the covariance function $R_u(\tau) = \frac{1}{2}A^2 \cos(\omega_1 \tau)$

Lösningar/svar

8.1

We have $\varphi(k) = (u_1(k) \ u_2(k))^T$, with $u_1(k) = K$ and $u_2(k) = L$. We assume the number of data to be N . This gives the $(N|2)$ matrix:

$$\Phi = \begin{bmatrix} K & L \\ \vdots & \vdots \\ K & L \end{bmatrix}$$

and

$$\Phi^T \Phi = \begin{bmatrix} NK^2 & NKL \\ NKL & NL^2 \end{bmatrix}$$

We then see that $\det(\Phi^T \Phi) = 0$ and hence the LS method can not be used. It is not possible to determine the parameters uniquely from this data set (which also should be intuitively clear).

8.2

The predictor for the ARX model is $\hat{y}(k) = \varphi^T(k)\theta$ where $\varphi(k) = (-y(k-1) \ u(k-2))^T$ and $\theta = (a \ b)^T$. The least squares estimate is

$$\begin{aligned} \hat{\theta} &= \left[\sum_{k=1}^N \varphi(k) \varphi^T(k) \right]^{-1} \sum_{k=1}^N \varphi(k) y(k) \\ &= \begin{bmatrix} \sum_{k=2}^{102} y^2(k-1) & -\sum_{k=2}^{102} y(k-1)u(k-1) \\ -\sum_{k=2}^{102} y(k-1)u(k-1) & \sum_{k=2}^{102} u^2(k-1) \end{bmatrix}^{-1} \begin{bmatrix} -\sum_{k=2}^{102} y(k-1)y(k) \\ \sum_{k=2}^{102} u(k-1)y(k) \end{bmatrix} \\ &= \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4.5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.875 \\ 0.125 \end{bmatrix} \end{aligned}$$

8.3

We have that

$$\begin{aligned} \text{cov}(\hat{\theta}) &= \lambda \left[\sum_{k=1}^N \varphi(k) \varphi^T(k) \right]^{-1} = \frac{\lambda}{N} \left[\frac{1}{N} \sum_{k=1}^N \varphi(k) \varphi^T(k) \right]^{-1} \rightarrow_{N \rightarrow \infty} \frac{\lambda}{N} [E\{\varphi(k) \varphi^T(k)\}]^{-1} \\ &= \frac{\lambda}{N} (\bar{R})^{-1} \end{aligned}$$

With $\varphi(k) = (u(k) \ u(k-1))^T$ we get

$$\bar{R} = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix}$$

and as $N \rightarrow \infty$

$$\text{cov}(\hat{\theta}) = \frac{\lambda}{N} \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix}^{-1}$$

Hence,

$$\text{var}(\hat{b}_o) = \text{var}(\hat{b}_1) = \frac{\lambda}{N} \frac{R_u(0)}{R_u^2(0) - R_u^2(1)}$$

b) It is seen directly (note that $R_u(0) \geq |R_u(\tau)|$) that the variances are minimized for $R_u(0) = 1$ and $R_u(1) = 0$. One example of a signal that fulfils this condition is white noise with unit variance.

8.4

The predictor is given by $\hat{y}(k) = \varphi^T(k)\theta$ where $\varphi(k) = (u(k-1) \ u(k-2))^T$ and $\theta = (b_1 \ b_2)^T$.

The least squares estimate is (we normalize with $\frac{1}{N}$ since we then get a feasible expression as $N \rightarrow \infty$)

$$\hat{\theta} = \frac{1}{N} \left[\frac{1}{N} \sum_{k=1}^N \varphi(k) \varphi^T(k) \right]^{-1} \sum_{k=1}^N \varphi(k) y(k)$$

As $N \rightarrow \infty$

$$\hat{\theta}_\infty = (\bar{R})^{-1} E\{\varphi(k)y(k)\}$$

where $\bar{R} = E\{\varphi(k)\varphi^T(k)\}$ cf the previous problem.

$$\hat{\theta} = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix}^{-1} \begin{bmatrix} R_{yu}(1) \\ R_{yu}(2) \end{bmatrix}$$

Since $u(t)$ is white noise $R_u(1) = 0$ and

$$R_{yu}(1) = E y(k) u(k-1) = E\{[b_1 u(k-1) + b_2 u(k-2) + e(k)] u(k-1)\} = b_1 R_u(0)$$

$$R_{yu}(2) = E y(k) u(k-2) = E\{[b_1 u(k-1) + b_2 u(k-2) + e(k)] u(k-2)\} = b_2 R_u(0)$$

the estimate converges to

$$\hat{\theta}_\infty = \begin{bmatrix} 1/R_u(0) & 0 \\ 0 & 1/R_u(0) \end{bmatrix} \begin{bmatrix} R_u(0)b_1 \\ R_u(0)b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

which is expected since we have a FIR model (which could be interpreted as a linear regression model) and model structure is correct.

b) We first calculate

$$\begin{aligned}\frac{1}{N} \sum_{k=1}^N \varphi(k) \varphi^T(k) &= \frac{1}{N} \begin{bmatrix} \sum_{k=1}^N u^2(k-1) & \sum_{k=1}^N u(k-1)u(k-2) \\ \sum_{k=1}^N u(k-1)u(k-2) & \sum_{k=1}^N u^2(k-1) \end{bmatrix} \\ &= \frac{1}{N} \begin{bmatrix} N-1 & N-2 \\ N-2 & N-2 \end{bmatrix} = \begin{bmatrix} \frac{N-1}{N} & \frac{N-2}{N} \\ \frac{N-2}{N} & \frac{N-2}{N} \end{bmatrix}\end{aligned}$$

and we see that

$$\bar{R} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is singular. This means that a step gives too poor excitation of the system, asymptotically the matrix (which should be inverted in the least squares method) becomes singular.

8.5

In this case $\varphi(k) = (u(k-1))$ and $\theta = b$. Asymptotically in N we have

$$\hat{\theta}_{\infty} = [E\{\varphi(k)\varphi^T(k)\}]^{-1} E\{\varphi(k)y(k)\}$$

For this simple predictor we have:

$$\begin{aligned}E\{\varphi(k)\varphi^T(k)\} &= Eu^2(k-1) = R_u(0) \\ E\{\varphi(k)y(k)\} &= Ey(k)u(k-1) = R_{yu}(1)\end{aligned}$$

and by using the system generating the data

$$R_{yu}(1) = Ey(k)u(k-1) = E\{[b_1u(t-1)+b_2u(k-1)+e(k)]u(k-1)\} = b_1R_u(0)+b_2R_u(1)$$

which gives

$$\hat{\theta}_{\infty} = \hat{b}_{\infty} = b_1 + b_2 \frac{R_u(1)}{R_u(0)}$$

a) If $u(k)$ is white noise $R_u(1) = 0$ and we get $\hat{b}_{\infty} = b_1$.

b) For $u(k)$ being a sinusoid, $\hat{b}_{\infty} = b_1 + b_2 \cos w_1$