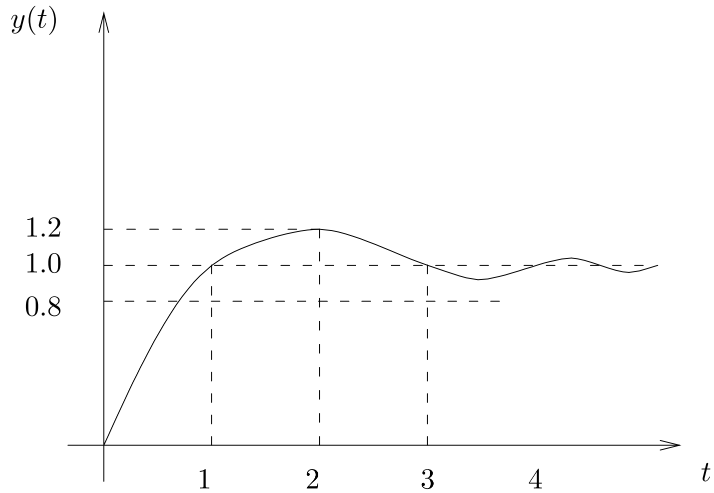
3 Räkneövningar till kapitel 3 i kompendiet

3.1

If the input to a certain system is a unit step, then the output will be according to Figure 1.

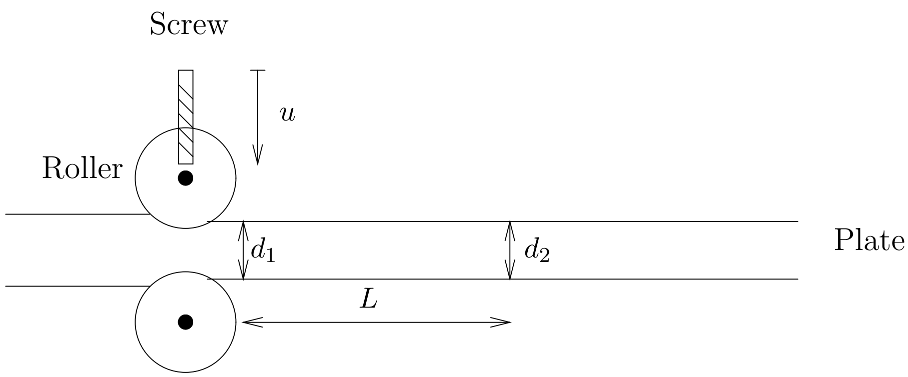


Figur 1: Step response for 1.1

Assume that the input instead is an impulse. What will the output be at time $t = 2$. Justify.

3.2

Consider the simple model of the roller depicted in Figure 2a.

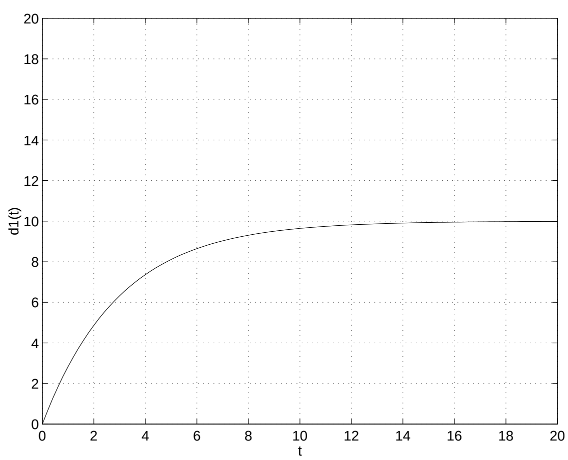


Figur 2: Picture of the system in 1.2

To obtain a simple model we describe the relationship between the position of the screw and the thickness of the sheet d_1 directly after the rollers as a first order transfer function

$$G_v(s) = \frac{\beta}{1 + sT}$$

To determine the constants β and T we register the effect of a sudden change in the position of the screw. Calculated in suitable units the change can be viewed as a unit step. In the Figure 3b the resulting thickness profile $d_1(t)$ is shown.



Figur 3: Step response for the system.

In production the thickness cannot be measured directly behind the rollers for practical reasons, and instead the thickness $d_2(t)$ is measured L length units after the rollers. Find the transfer function from the position of the screws to the thickness d_2. The sheet moves with speed V.

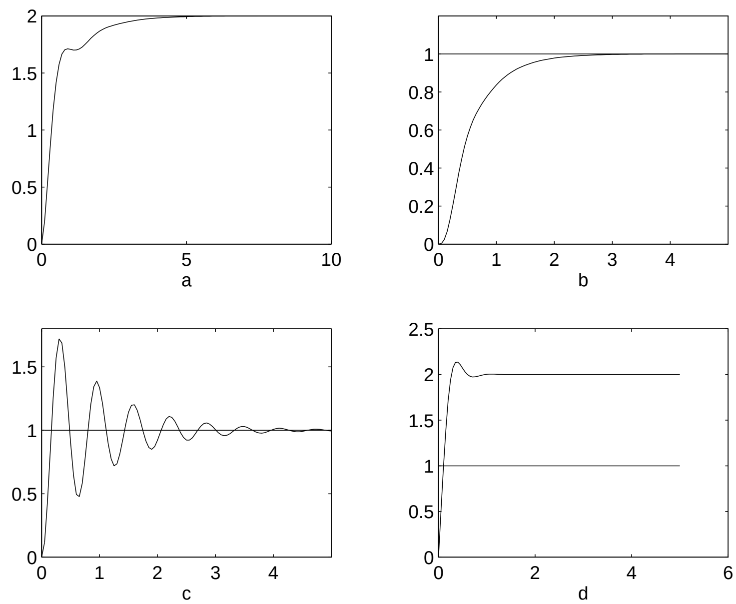
3.3

Four step responses are shown below. Combine the step responses with the correct transfer function among the alternatives below. Justify!

$$G_1(s) = \frac{100}{s^2 + 2s + 100} \qquad G_2(s) = \frac{1}{s+2}$$

$$G_3(s) = \frac{10s^2 + 200s + 2000}{(s+10)(s^2 + 10s + 100)} \qquad G_4(s) = \frac{100}{s^2 + 10s + 100} \cdot \frac{2}{s+2}$$

$$G_5(s) = \frac{100}{s^2 + 10s + 100} + \frac{2}{s+2} \qquad G_6(s) = \frac{100}{s^2 - 10s + 100} \cdot \frac{1}{s+2}$$



Figur 4: Step responses for 1.3

3.4

A system has the transfer function

$$G(s) = \frac{e^{-2s}}{s(s+1)}$$

What is the output (after all transients have vanished) when the input is

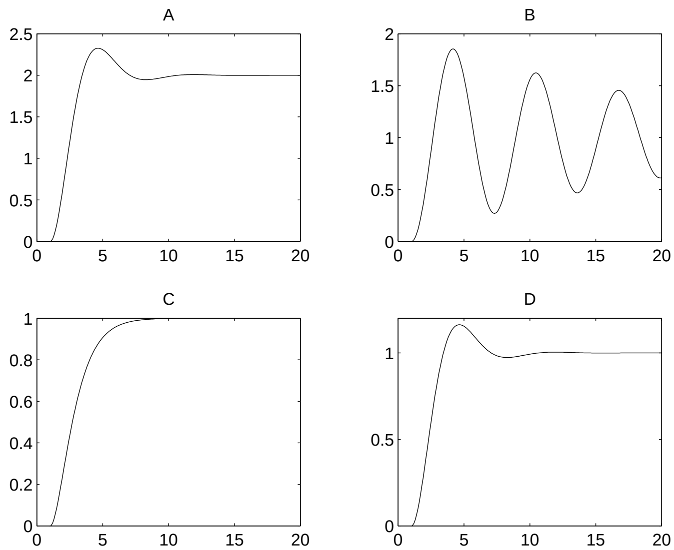
$$2\sin(2t - 1/2)$$

3.5

In Figure 3.5 step responses for the following systems are plotted. Combine the systems and the step responses. Justify your answer!

$$G_1(s) = \frac{2}{s^2 + s + 1} \qquad G_2(s) = \frac{1}{s^2 + s + 1}$$

$$G_3(s) = \frac{1}{s^2 + 0.1s + 1} \qquad G_4(s) = \frac{1}{s^2 + 2s + 1}$$



Figur 5: Step responses for 1.5

3.6 This excercise is suitable to solve by computer (matlab)

Consider the systems

$$G_A(s) = \frac{1}{s^2 + 2s + 1}$$

$$G_B(s) = \frac{1}{s^2 + 0.4s + 1}$$

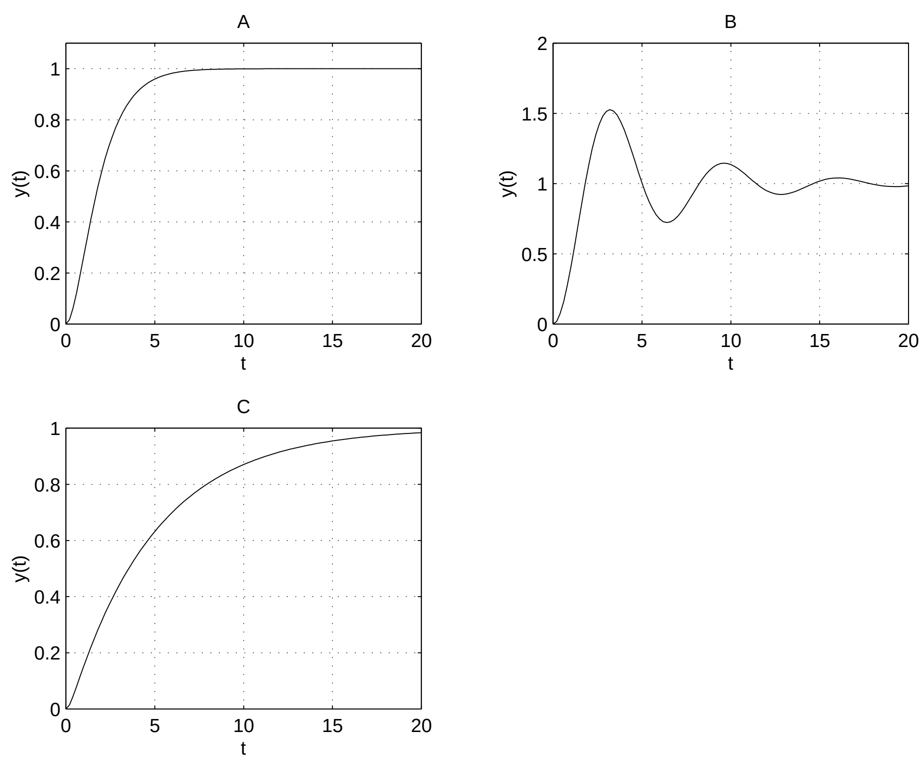
and

$$G_C(s) = \frac{1}{s^2 + 5s + 1}$$

a) The step responses of the three systems are given in Figure 3.6. Find t_r (stigtid), t_5 (insvängningstid) and M (relativ översläng) for the three step responses.

b) Compute the poles of the systems $G_A(s), G_B(s)$ and $G_C(s)$ respectively.

c) How is the location of the poles related to the properties of the step responses?



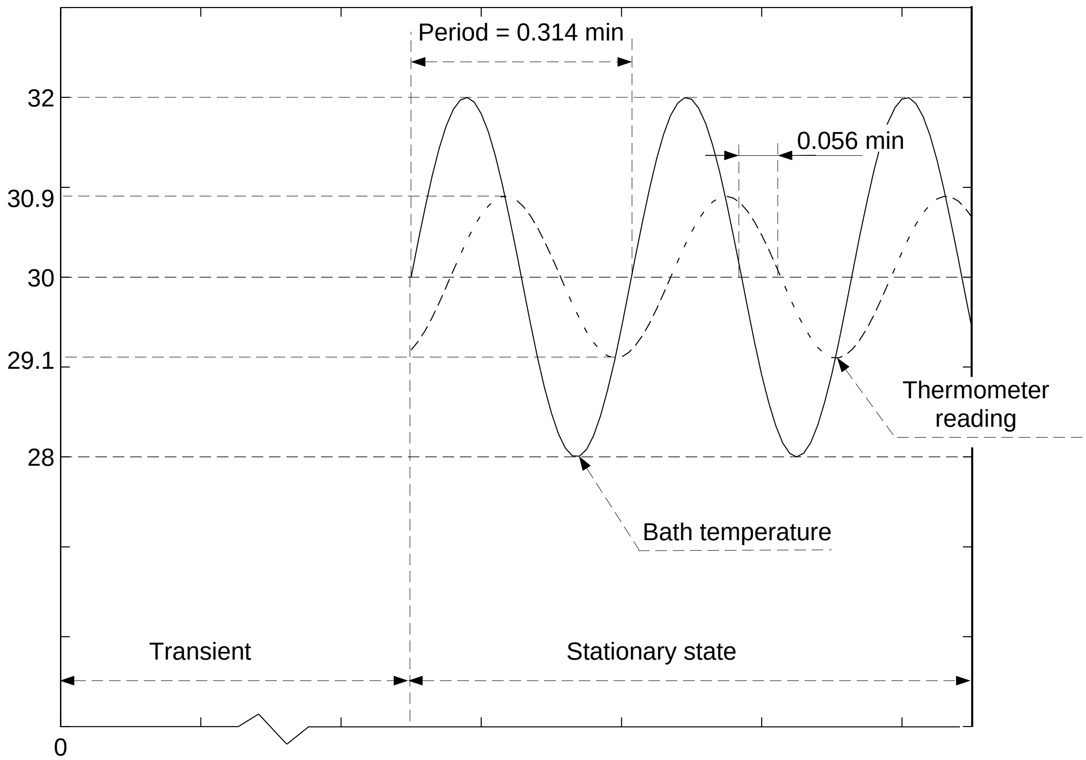
Figur 6: Step responses for 1.6

3.7

A mercury thermometer can be described with high accuracy as the following first order linear time invariant dynamic system The thermometer is modeled as the following first order linear time invariant dynamic system with

$$\frac{Y(s)}{U(s)} = G(s) = \frac{a}{s+b}$$

where input ($U(s)$, $u(t)$) is the real temperature and the output ($Y(s)$, $y(t)$)is the thermometer reading. In order to determine the transfer function of the thermometer, it is placed in a liquid where the temperature is varied as a sinusoid. The obtained result is shown in Figure 7.



Figur 7: Sinusoidal response for 1.7

Find the transfer function of the thermometer.

3.8

Consider the following systems

a)

$$G(s) = \frac{2}{s+1}$$

b)

$$G(s) = e^{-2s}$$

c)

$$G(s) = \frac{1}{s^3 + 1}$$

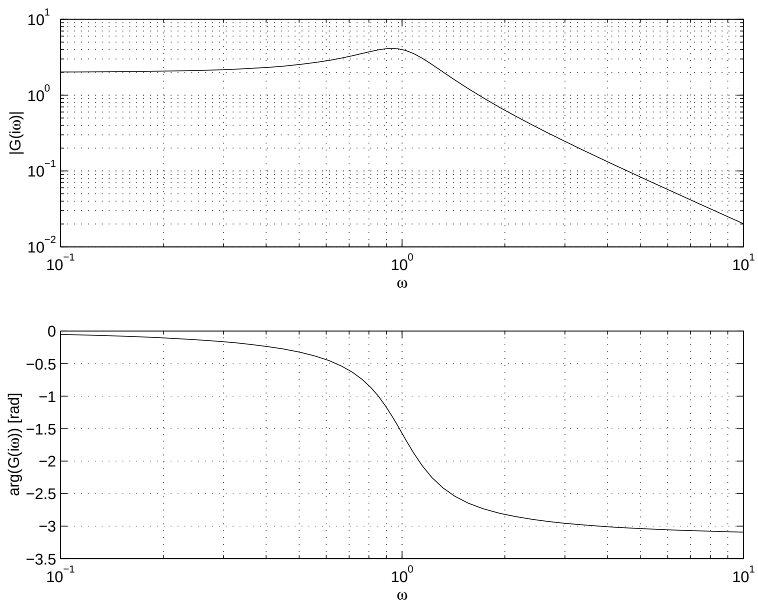
Let the input be $u(t) = \sin t$. Compute, in the cases where it is meaningful, the output when the transients have disappeared.

3.9

I Figur 8 visas ett bodediagrammet för ett system (notera att ett bodediagram kan erhållas experimentiellt via mätningar)

a) Antag att insignalen till systemet är en konstant u_0. Vad blir utsignalen stationärt? Vad är systemets statiska förstärkning?

b) Antag nu att systemet påverkas av en periodisk insignal. Insignalen kan skrivas $u(t) = 2\sin(t-1) + \sin 2t$. Vad blir utsignalen från systemet?



Figur 8: Bodediagram för 1.9

3.10

Antag att vi vill studera impulsssvar och stegsvar för systemen med överföringsfunktioner givna nedan:

$$G_1(s) = \frac{s+3}{s+7}$$
$$G_2(s) = \frac{s+1}{s^2+5s+1}$$
$$G_3(s) = \frac{1}{(s+5)(s-5)}$$

a) Är respektive system insignal-utsignalstabilt?

b) Vad blir utsignalen för respektive system stationärt (d v s då $t \to \infty$) då insignalen är ett steg?

c) Hur beter sig utsignalen från respektive system i början av stegsvarsförloppet (d v s då $t \to 0^+$)?

Lösningar/svar

3.1

The Laplace transform of the step response is given by

$$Y(s) = G(s)\frac{1}{s}$$

where $G(s)$ is the system transfer function. The impulse response $g(t)$ is the inverse Laplace transform of the transfer function, which implies

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{sY(s)\} = \dot{y}(t)$$

i.e. the impulse response is the derivative of the step response. The figure shows that the derivative of the step response is zero at $t = 2$.

Answer: The impulse response is zero at the time $t = 2$.

3.2

The inverse Laplace transform gives the step response

$$d_1(t) = \mathcal{L}^{-1}\left\{\frac{\beta}{1+sT}\cdot\frac{1}{s}\right\} = \beta(1-e^{-t/T})$$

When $t \to \infty$ we get

$$d_1(t) \to \beta$$

The figure gives

$$\beta = 10$$

At the time $t = T$, the system time constant, the step response has reached 63% of the final value, i.e.

$$d_1(T) = 0.63 \cdot 10$$

The figure gives

$$T = 3$$

which gives the total transfer function

$$G_v(s) = \frac{10}{1+3s}$$

If we measure the signal $d_2(t)$ we introduce an additional time delay of $\frac{L}{V}$ time units. The total transfer function then becomes

$$G_v(s) = \frac{10e^{-\frac{L}{V}s}}{1+3s}$$

Answer:

$$G_v(s) = \frac{10e^{-\frac{L}{V}s}}{1+3s}$$

3.3

G_6 can be excluded since the system is unstable. G_2 can alse be excluded since the systen has static gain 0.5. G_1 and G_4 have unity static gain. G_1 has lightly damped complex poles. G_4 is dominated by a real pole. This gives the combinations G_1 – c and G_4 – b. $|G_3(0)| = |G_5(0)| = 2$ and they both have the same complex poles, but G_5 has a slower real pole. This gives G_3 – d and G_5 – a.

3.4

$$
\begin{aligned}
G(s) = \frac{e^{-2s}}{s(s+1)}, u(t) = 2\sin(2t - 1/2) \\
\Rightarrow y(t) = 2|G(i2)|\sin(2t - 1/2 + \arg G(i2))
\end{aligned}
$$

where

$$
\begin{aligned}
|G(i2)| &= \frac{1}{2\sqrt{2^2+1}} = \frac{1}{2\sqrt{5}} \\
\arg G(i2) &= -4 - \frac{\pi}{2} - \arctan 2
\end{aligned}
$$

3.5

G_1: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sG_1(s)\frac{1}{s} = 2 \Rightarrow G_1 \leftrightarrow$ step response A.

G_4: has only real poles, hence the step response will not be oscillative or have any overshoot. $\Rightarrow G_4 \leftrightarrow$ step response C.

G_2 and G_3: The poles of G_2 are $-0.5 \pm 0.87i$ and the poles of G_3 are $-0.05 \pm 0.99i$. The poles of G_3 is dominating by the imaginary part (it has a much larger angle from the negative real axis) and the step response should be more oscillative than for G_2.
$\Rightarrow G_2 \leftrightarrow$ step response D.
$\Rightarrow G_3 \leftrightarrow$ step response B.

3.6

a) Enter the systems.

```
>> GA=tf(1,[1 2 1]);
>> GB=tf(1,[1 0.4 1]);
>> GC=tf(1,[1 5 1]);
```

Compute and plot the step response.

```
>> step(GA); grid
```

The systems $G_B(s)$ and $G_C(s)$ can be simulated in a similar way. The values of t_r, t_5 and M for the different step responses can be found using the `ginput` command.

b) Compute the poles.

```
>> pole(GA)

ans =

    -1
    -1
```

The other systems are handled in the same way.

c) The results from a) and b) can be summarized in the following table. It is important to note that the values generated by `rtplot` is slightly affected by the choice of simulation parameters.

System	t_r	t_5	M	poles
G_A	3.3	5.7	0%	$-1, -1$
G_B	1.2	14.7	53%	$-0.2 \pm i0.98$
G_C	10.6	15.5	0%	$-4.8, -0.2$

Using this table we can draw the following conclusions. (i): The speed of the step response (mainly) depends on the distance between the poles and the origin. Poles further away from the origin give a faster step response and shorter rise time. (ii): The damping of the system depends on the relationship between the imaginary part and the real part of the poles. Poles with large imaginary part relative to the real part give a poorly damped (oscillatory) step response.
Remark: We see that even though the distance to the origin is nearly the same in system G_A and G_B the rise time is almost 3 times faster in system B. Note that speed is not only rise time, even the solution time should be considered. Look at the following system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0}$$

The poles of this system are given by $s = \omega_0(-\zeta \pm i\sqrt{1-\zeta^2})$ where $\cos\phi = \zeta$. The parameter ζ is called relative damping and $0 \leq \zeta \leq 1$. We see clearly that the rise time is faster when ζ is small but when ζ is small the solution time is big!

3.7

Introduce the notations $T_v(t)$ and $T_t(t)$ for the actual temperature and the measured temperature, respectively. Divide the temperature into its mean value and its variations:

$$T_v(t) = u_0 + u(t)$$

and

$$T_t(t) = y_0 + y(t)$$

where $u_0 = y_0 = 30^\circ\ C$.
The thermometer is modeled as the following first order linear time invariant dynamic system with

$$\frac{Y(s)}{U(s)} = G(s) = \frac{a}{s+b}$$

Since

$$u(t) = A\sin(\omega t)$$

it follows that after the transients have vanished (i.e. in stationarity)

$$y(t) = \mid G(i\omega) \mid A\sin(\omega t + \phi)$$

where

$$\phi = \arg(G(i\omega)) = -\arctan(\omega/b)$$

From the relationship $\omega = 2\pi/T$ and from the figure the following is obtained:

1. $\omega = \frac{2\pi}{0.314\cdot 60}$ rad/s $= 0.33$ rad/s.
2. $\phi = \frac{-0.056}{0.314} \cdot 2\pi$ rad $= -1.12$ rad.
3. $\mid G(i\omega) \mid = \frac{0.9}{2.0} = 0.45$

Hence

$$\tan(\phi) = -\frac{\omega}{b} \Rightarrow b = \frac{0.33}{2.066} = 0.16$$

and

$$\mid G(i\omega) \mid = \frac{a}{\sqrt{\omega^2 + b^2}} \Rightarrow a = 0.16$$

Answer:

$$G(s) = \frac{0.16}{s + 0.16}$$

(Note that $G(0) = 1$, hence the thermometer gives the correct temperature reading when the input temperature is a constant.)

3.8

a)

$$y(t) = |G(i\omega)|\sin(t + \arg(G(i\omega))) = [\omega = 1] = \sqrt{2}\sin(t - \pi/4)$$

b)

$$y(t) = \sin(t + \arg(e^{-2i})) = \sin(t-2)$$

c) The system is unstable, poles in $s = e^{i(\pi/3 + k2\pi/3)}, k = 1, 2, 3$.

Remark:
Consider the following characteristic equation

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = 0$$

A necessary, but **not** sufficient, condition for all poles to be strictly in the left half plane (that is, the system is stable) is that all coefficients $\{a_i\}_{i=1}^n$ must be strictly positive. If any coefficient is zero or negativ the system not stable. For polynomials of degree two this requirement is both necessary and sufficient.

3.9

a) $y(\infty) = 2u_0$, statiska förstärkningen G(0)=|G(0)|=2.

b) $y(t) = 8\sin(t - 2.6) + 0.6\sin(2t - 2.8)$

3.10

a) System 1 och 2: stabilt. System 3: ej stabilt.
(Notera att om alla poler har strikt negativ realdel så är systemet både insignal-utsignalstabilt och asymptotiskt stabilt)

b) System 1: $\Rightarrow$ $y(\infty)$=3/7
System 2: $\Rightarrow$ $y(\infty)$=1
System 3: $\Rightarrow$ $y(\infty) = \infty$ (OBS Slutvärdesteoremet kan inte användas!)

•

c) System 1: $y(0^+)$=1. Övriga system $y(0^+)$=0.