Review: Duality

"Canonical forms":

For a minimization problem: For a maximization problem:

$$\min_x c^T x \quad \max_y b^T y$$

$$Ax \geq b \quad (P) \quad A^T y \leq c \quad (D)$$

$$x \geq 0 \quad y \geq 0$$

Problems (P) and (D) are the dual to each other.

Standard form:

$$\min_x c^T x \quad \max_y b^T y$$

$$Ax = b \quad (P) \quad A^T y \leq c \quad (D)$$

$$x \geq 0$$

General rules

<table>
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<th>constraint</th>
<th>variable</th>
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<tbody>
<tr>
<td>ineq. as in canonical form</td>
<td>$\geq 0$</td>
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<tr>
<td>ineq. reversed from canonical form</td>
<td>$\leq 0$</td>
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<tr>
<td>equality</td>
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Example:

$$\max 6x_1 + x_2 + x_3 \quad \min y_1 + 9y_2 + 5y_3$$

$$4x_1 + 3x_2 - 2x_3 = 1 \quad 4y_1 + 6y_2 + 2y_3 \geq 6$$

$$6x_1 - 2x_2 + 9x_3 \geq 9 \quad 3y_1 - 2y_2 + 3y_3 \leq 1$$

$$2x_1 + 3x_2 + 8x_3 \leq 5 \quad -2y_1 + 9y_2 + 8y_3 = 1$$

$$x_1 \geq 0, x_2 \leq 0, x_3 \text{ unrestricted} \quad y_1 \text{ unrestricted, } y_2 \leq 0, y_3 \geq 0.$$
Standard form:

\[
\begin{align*}
\min_{x} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\] (P)

\[
\begin{align*}
\max_{y} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c
\end{align*}
\] (D)

**Theorem.** (Weak duality.) If \(x, y\) are feasible for (P) and (D) then \(b^T y \leq c^T x\).

**Corollary 1.** Primal (dual) problem unbounded \(\Rightarrow\) dual (primal) problem infeasible.

**Corollary 2.** If \(x, y\) are feasible for primal and dual problem and \(c^T x = b^T y\) then \(x, y\) are optimal.

**Theorem.** (Strong duality.) (P) has an optimal solution \(x_*\) if and only if (D) has an optimal solution \(y_*\) and \(c^T x_* = b^T y_*\).

The simplex multipliers provides the dual solution at the optimum for the simplex algorithm: \(y = B^{-T} c_B\) solves the dual problem if \(B\) is the basis of the optimal solution of the primal problem.

**Complementary slackness.** Let \(x_*, y_*\) solve (P) and (D). Strong duality: \(x_*^T c = b^T y_* = (Ax_*)^T y_* = x_*^T A^T y_*\), so

\[
0 = x_*^T (c - A^T y_*) = \sum_{i=1}^{n} x_{*,i} (c - A^T y_*)_i.
\]

Either \(x_{*,i} > 0\) or \((c - A^T y_*)_i > 0\) not both!
**Sensitivity:** The dual solution yields cost sensitivity for changes in $b$

$$\min_{x} \zeta = c^T x$$

$$Ax = b$$

$$x \geq 0$$

$x_*$: nondegenerate optimal basic solution,

$$x_{*B} = B^{-1} b > 0$$

$$x_{*N} = 0$$

Small changes $\delta b$ in $b$ does not change optimal basis so the change $\delta x_*$ in $x_*$ is

$$\delta x_{*B} = B^{-1} \delta b,$$

$$\delta x_{*N} = 0$$

and the change $\delta \zeta$ in the objective $z$ will be

$$\delta \zeta = c^T \delta x_* = c^T_B \delta x_{*B} + c^T_N \delta x_{*N} = c^T_B B^{-1} \delta b = y^*_T \delta b$$