1. A production plant manufactures its final product from raw materials A, B, C, and D. The amount of available raw material for the next production period is

<table>
<thead>
<tr>
<th>Raw material</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Availability</td>
<td>24</td>
<td>27</td>
<td>24</td>
<td>20</td>
</tr>
</tbody>
</table>

The plant uses two different processes. Process 1 produces each hour 3 units of the final product using 2, 3, 3, and 4 units of raw materials A, B, C, and D, whereas process 2 each hour produces 6 units of the final product using 4, 3, 6, and 1 units of the raw materials.

(a) Set up and find all solutions to the linear program that allocates time for the two processes in order to maximize the output of the plant. (4p)

(b) Assume now that the availability of raw material A increases with 10% while the availability for the other raw materials is unchanged. Then assume the same scenario for materials B, C, and D. What will the difference in the maximal number of manufactured products be in each of these four scenarios? (2p)

2. Let

\[ f(x_1, x_2) = x_1^2 + x_1x_2 + x_2 - 2x_1. \]

(a) Is \( f \) convex? Motivate your answer. (2p)

(b) Suppose that \( g \) is a convex function on \( \mathbb{R}^n \). Show that the set of global minimizers is a convex set. (3p)

3. A standard choice of algorithm for solving the unconstrained optimization problem \( \min_{x \in \mathbb{R}^n} f(x) \) is a quasi-Newton algorithm with a secant approximation of the Hessian.

(a) What is the advantage of this method compared to (i) Newton’s method and (ii) the steepest-descent method? (2p)

(b) The secant condition for a Hessian approximation \( B_{k+1} \) is

\[ B_{k+1}s_k = y_k, \]

where

\[ s_k = x_{k+1} - x_k, \]

\[ y_k = \nabla f(x_{k+1}) - \nabla f(x_k). \]

Why do we want \( B_{k+1} \) to satisfy the secant condition? (2p)
(c) Show that the curvature condition $s_k^T y_k > 0$ is necessary for the Hessian approximation $B_{k+1}$ to be positive definite.

(d) Show that the curvature condition is always satisfied for strictly convex functions. 

Hint: Strictly convex, differentiable functions on $\mathbb{R}^n$ satisfy

$$f(y) > f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in \mathbb{R}^n, x \neq y$$

4. Consider the iterative process

$$x_{k+1} = \frac{1}{2} \left( x_k + \frac{a}{x_k} \right),$$

where $a > 0$. Assume that the process converges.

(a) To what does it converge? (2p)

(b) Show that the convergence rate is 2 (quadratic). (4p)

5. From physical arguments there is reason to believe that the behavior of a process is modelled by

$$y(t) = a + t^b$$

when time $t$ grows, but the parameters $a$ and $b$ are unknown. The process has been measured at $t_j, j = 1, 2, 3$, with the result

<table>
<thead>
<tr>
<th>$j$</th>
<th>$t_j$</th>
<th>$y_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2.4</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

(a) Use Gauss–Newton’s method to determine suitable values of $(a, b)$ in the least squares sense with $(1.5, 2.2)$ as initial guess. Compute the solution after one iteration. (4p)

(b) The residual $r$ is measured in the norm $\| \cdot \|_2$, defined by $\| u \|_2^2 = \sum_{j=1}^n u_j^2$ for a vector $u$ of length $n$. Compute the initial residual and the residual after one iteration. Does the iteration seem to converge? (2p)

6. Simulated annealing and the genetic algorithm are two methods to compute a global optimum of an objective function.

(a) What is the condition for a new position in the search space to be accepted in simulated annealing? (2p)

(b) Discuss how this condition depends on the parameters in the method. (1p)

(c) Describe how the new generation is selected with the roulette wheel in the genetic algorithm. (2p)

(d) Describe crossover in the genetic algorithm. (1p)

Good luck!

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