Basic solutions

An LP in standard form:

$$\begin{align*}
\text{min } & \quad c^T x \\
\text{subject to } & \quad Ax = b \\
& \quad x \geq 0,
\end{align*}$$

Matrix $A$ is $m$-by-$n$. Assume linearly independent rows, i.e., $\text{rank } A = m$.

Partition $A$ by columns: $A = (a_1, \ldots, a_n)$, where $a_i$ is a column vector of length $m$. Note that $Ax = \sum_{i=1}^n a_i x_i$.

Let $I_B \subset I = \{1, \ldots, n\}$ and $I_N = I \setminus I_B$. Split the sum:

$$Ax = \sum_{i \in I_B} a_i x_i + \sum_{i \in I_N} a_i x_i$$

so the columns $\{a_i\}_{i \in I_B}$ are linearly independent.

Example:

$$Ax = \begin{pmatrix} 0 & 0 & 3 & 4 & 1 & 4 \\ 1 & -2 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$$

$Ax = b$ for many $x$, for instance

(i) $x = (2, 0, 1, 2, 1, 0)^T$. One possible partition: $x_B = (x_1, x_3)^T = (2, 1)^T$ basic variables, and $x_N = (x_2, x_4, x_5, x_6)^T = (0, 2, 1, 0)^T$ nonbasic. There are several other possible partitions. Note: $x = (2, 0, 1, 2, 1, 0)^T$ is a solution (to $Ax = b$) but not a basic solution since $x_N \neq 0$.

(ii) $x = (0, -3/2, 4, 0, 0, 0)$ is a basic solution but not feasible.

(iii) $x = (0, 0, 3, 0, 3, 0)^T$ is a basic feasible solution.

(iv) $x = (0, 0, 0, 0, 3)^T$ is a degenerate basic feasible solution.