Assignment 3: Portfolio Optimization

Submit a written report containing a short presentation of the problem, results, discussion, and source code. Remember to answer all question and comment your results. You may work alone or in groups of two (not three). The absolute deadline for submitting this years assignments is 2005-11-04. You will obtain a 1/2-point bonus point on the nal exam if a correct solution for this assignment is handed in at the latest 2005-10-20.

The Markowitz framework for portfolio selection

The return of investment instruments such as publicly-traded stocks are unknown at the time when the investment is made. Typically, there is a tradeoff between risk and return: a higher return is associated with a higher risk. A quantitative study of such tradeoffs is made particularly simple using an analysis from the 1950’s due to Markowitz assuming that the returns of the investments are random variables with a known multivariate normal distribution.

An asset is an investment instrument that can be bought or sold. The rate of return of the asset is the number \( r \) satisfying \( X_1 = (1 + r)X_0 \), where \( X_0 \) and \( X_1 \) are the prices of the asset at purchase and selling, respectively. As an example, the rate of return from deposits in a bank account is the interest rate. Assume that an investor wants to select a portfolio of \( n \) possible assets. The investor puts the fraction \( w_i \) of available funds into asset \( i \) with rate of return \( r_i, i = 1, \ldots, n \). We assume that all available funds are invested, so \( \sum_{i=1}^{n} w_i = 1 \). Thus, rate of return for the portfolio will be \( r = \sum_{i=1}^{n} r_i w_i \).

The situation when a weight \( w_i \) is negative corresponds to a short selling of the asset, that is, the investor buys the asset, sells it to someone else, and use the amount thus received to invest in the other assets. Short selling can significantly increase the total return of the portfolio at the price of a substantially increased risk. When short selling is not allowed, we require that \( w_i \geq 0 \), whereas there are no constraints on \( w_i \) when unlimited short selling of the asset is allowed.

The rates of returns are often not known in advance. We assume in the Markowitz framework that we possess estimates of their expected values \( \tilde{r}_i = \mathbb{E}(r_i) \) and their covariances \( \sigma_{ij} = \mathbb{E}((r_i - \tilde{r}_i)(r_j - \tilde{r}_j)) \). These estimates can for instance be obtained by analysis of previous performance of the assets. The properties of the expectation value yields that the expected rate of return and the variance of the portfolio will be \( \tilde{r} = \sum_{i=1}^{n} r_i w_i \) and \( \sigma = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \).

We will consider portfolios of five assets with yearly expected rates of returns and covariances according to Table 1. Note that any matrix of covariances is symmetric and positive semidefinite.

To do

1. Formulate the linear program of maximizing the expected return of the portfolio when short selling is not allowed. Solve the linear program by “inspection” (that is, do not use software or hand calculation). What is the variance of the portfolio?

2. Formulate the quadratic program of minimizing the variance of the portfolio (short selling allowed). Set up the KKT system and solve it using Matlab. Report the computed weights and the variance of corresponding portfolio and compare with 1.

Note that Matlab can handle block matrices. Assume that \( A \) is an \( n \)-by-\( n \) matrix and \( b \) a column vector of dimension \( n \), and that these can be blocked as

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},
\]

where
Table 1: Covariances and rates of returns for five assets. Note that the numbers in the table should be multiplied with $10^{-2}$. For instance, $\bar{r}_1 = 0.151$ and $\sigma_{11} = 0.023$. These values are borrowed from Example 6.11 of Chapter 6 in Luenberger: Investment Science, Oxford Univ. Press (1997).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Covariances ($\times 10^2$)</th>
<th>$\bar{r}_i$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.30 0.93 0.62 0.74 −0.23</td>
<td>15.1</td>
</tr>
<tr>
<td>2</td>
<td>1.40 0.22 0.56</td>
<td>12.5</td>
</tr>
<tr>
<td>3</td>
<td>1.80 0.78</td>
<td>14.7</td>
</tr>
<tr>
<td>4</td>
<td>3.40 −0.56</td>
<td>9.02</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>2.60 17.68</td>
</tr>
</tbody>
</table>

- matrix $A_{11}$ has as many rows as matrix $A_{12}$ and matrix $A_{21}$ has as many rows as matrix $A_{22}$,
- matrix $A_{11}$ has as many columns as matrix $A_{21}$ and matrix $A_{12}$ has as many columns as matrix $A_{22}$;
- both $b_1$ and $b_2$ are column vectors.

Then Matlab matrices $A$ and $b$ can be specified as

$$A = [A_{11}, A_{12}; A_{21}; A_{22}];$$
$$b = [b_1; b_2];$$

where $A_{11}, A_{12}, A_{21}, A_{22}, b_1,$ and $b_2$ are previously defined matrices with dimensions that match as above. Note that elements on the same row are separated with , (a separating space is also OK), whereas a new row is indicated by ; (inserting a new row by pressing the return key is also OK).

3. An extremely risk averse investor would choose the strategy in 2 whereas an extremely risk preferring investor would choose the strategy in 1. Most investors choose a strategy in between these extremes. For instance, if an investor requires a particular expected rate of return $\bar{r}$, it makes sense to calculate the particular portfolio that meets that goal at the minimum variance. Formulate the quadratic program that minimizes the portfolio variance for a given expected rate of return of the portfolio. Formulate the problem both for the case when short selling is and is not allowed.

4. Let $\sigma$ and $\bar{r}$ be the portfolio variance and expected rate of return calculated as in 3 above. The efficient frontier is the set of all possible pairs $(\sigma, \bar{r})$. A good way of computing the efficient frontier is to solve, for a parameter $0 \leq \alpha \leq 1$, the quadratic program

$$\min \left( \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} - (1 - \alpha) \sum_{i=1}^{n} \bar{r}_i w_i \right)$$

subject to $\sum_{i=1}^{n} w_i = 1$, and, when short selling is not allowed,

$$w_i \geq 0, \text{ for } i = 1, \ldots, n.$$  \hspace{1cm} (1)

(a) Which cases do the extreme values $\alpha = 0$ and $\alpha = 1$ correspond to?

(b) Solve (1) for $\alpha = 0.05, 0.1, 0.15, \ldots, 1.0$ using quadprog, both for the case when short selling is and is not allowed. Plot and compare the efficient frontiers.