Review: Duality

Canonical forms:

For a minimization problem:

$$\min_x c^{\mathsf{T}} x \quad Ax \geq b \quad x \geq 0 \qquad \text{(P)}$$

For a maximization problem:

$$\max_y b^{\mathsf{T}} y \quad A^{\mathsf{T}} y \leq c \quad y \geq 0 \qquad \text{(D)}$$

Problems (P) and (D) are the dual to each other.

Standard form:

$$\min_x c^{\mathsf{T}} x \quad Ax = b \quad x \geq 0 \qquad \text{(P)}$$

$$\max_y b^{\mathsf{T}} y \quad A^{\mathsf{T}} y \leq c \qquad \text{(D)}$$

Duality

General rules

constraint	**variable**
ineq. as in canonical form	≥ 0
ineq. reversed from canonical form	≤ 0
equality	unrestricted

Example:

$$\begin{aligned} \max\ & 6x_1 + x_2 + x_3 \\ & 4x_1 + 3x_2 - 2x_3 = 1 \\ & 6x_1 - 2x_2 + 9x_3 \geq 9 \\ & 2x_1 + 3x_2 + 8x_3 \leq 5 \\ & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted} \end{aligned}$$

$$\begin{aligned} \min\ & y_1 + 9y_2 + 5y_3 \\ & 4y_1 + 6y_2 + 2y_3 \geq 6 \\ & 3y_1 - 2y_2 + 3y_3 \leq 1 \\ & -2y_1 + 9y_2 + 8y_3 = 1 \\ & y_1 \text{ unrestricted},\ y_2 \leq 0,\ y_3 \geq 0, \end{aligned}$$

Standard form:

$$\min_x c^\top x \quad Ax = b \quad x \geq 0 \qquad \text{(P)}$$

$$\max_y b^\top y \quad A^\top y \leq c \qquad \text{(D)}$$

Theorem. (Weak duality.) If x, y are feasible for (P) and (D) then $b^\top y \leq c^\top x$.
Corollary 1. Primal (dual) problem unbounded $\Rightarrow$ dual (primal) problem infeasible.
Corollary 2. If x, y are feasible for primal and dual problem and $c^\top x = b^\top y$ then x, y are optimal.

Theorem. (Strong duality.) (P) has an optimal solution x_* if and only if (D) has an optimal solution y_* and $c^\top x_* = b^\top y_*$.

The simplex multipliers provides the dual solution at the optimum for the simplex algorithm:

$y = B^{-\top} c_B$ solves the dual problem if B is the basis of the optimal solution of the primal problem.

Complementary slackness. Let x_*, y_* solve (P) and (D). Strong duality: $x_*^\top c = b^\top y_* = (Ax_*)^\top y_* = x_*^\top A^\top y_*$, so

$$0 = x_*^\top (c - A^\top y_*) = \sum_{i=1}^{n} x_{*,i} (c - A^\top y_*)_i.$$

$x_{*,i} > 0 \Rightarrow (c - A^\top y_*)_i = 0$; $(c - A^\top y_*)_i > 0 \Rightarrow x_{*,i} = 0$

Either $x_{*,i} > 0$ or $(c - A^\top y_*)_i > 0$ **not both**!

Sensitivity: The dual solution yields cost sensitivity for changes in b

$$\min_x \zeta = c^\mathsf{T} x$$
$$Ax = b$$
$$x \geq 0$$

x_*: nondegenerate optimal basic solution,

$$x_{*B} = B^{-1} b > 0$$
$$x_{*N} = 0$$

Small changes δb in b does not change optimal basis so the change δx_* in x_* is

$$\delta x_{*B} = B^{-1}\, \delta b,$$
$$\delta x_{*N} = 0$$

and the change $\delta\zeta$ in the objective z will be

$$\delta\zeta = c^\mathsf{T}\, \delta x_* = c_B^\mathsf{T}\, \delta x_{*B} + c_N^\mathsf{T}\, \delta x_{*N} = c_B^\mathsf{T} B^{-1}\, \delta b = y_*^\mathsf{T}\, \delta b$$