Review: Duality

Canonical forms:
For a minimization problem: For a maximization problem:

\[
\begin{align*}
\min_{x} & \quad c^T x \\
Ax & \geq b \quad \text{(P)} \\
x & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y} & \quad b^T y \\
A^T y & \leq c \quad \text{(D)} \\
y & \geq 0
\end{align*}
\]

Problems (P) and (D) are the dual to each other.

Standard form:

\[
\begin{align*}
\min_{x} & \quad c^T x \\
Ax & = b \quad \text{(P)} \\
x & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y} & \quad b^T y \\
A^T y & \leq c \quad \text{(D)}
\end{align*}
\]
Duality

General rules

<table>
<thead>
<tr>
<th>constraint</th>
<th>variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>ineq. as in canonical form</td>
<td>$\geq 0$</td>
</tr>
<tr>
<td>ineq. reversed from canonical form</td>
<td>$\leq 0$</td>
</tr>
<tr>
<td>equality</td>
<td>unrestricted</td>
</tr>
</tbody>
</table>

Example:

\[
\begin{align*}
\text{max } 6x_1 + x_2 + x_3 & \quad \text{min } y_1 + 9y_2 + 5y_3 \\
4x_1 + 3x_2 - 2x_3 = 1 & \quad 4y_1 + 6y_2 + 2y_3 \geq 6 \\
6x_1 - 2x_2 + 9x_3 \geq 9 & \quad 3y_1 - 2y_2 + 3y_3 \leq 1 \\
2x_1 + 3x_2 + 8x_3 \leq 5 & \quad -2y_1 + 9y_2 + 8y_3 = 1 \\
x_1 \geq 0, x_2 \leq 0, x_3 \text{ unrestricted} & \quad y_1 \text{ unrestricted, } y_2 \leq 0, y_3 \geq 0,
\end{align*}
\]
Duality

Standard form:

\[
\begin{align*}
\min_{x} & \quad c^T x \\
A x & = b \\
x & \geq 0
\end{align*}
\] (P)

\[
\begin{align*}
\max_{y} & \quad b^T y \\
A^T y & \leq c
\end{align*}
\] (D)

**Theorem.** (Weak duality.) If \(x, y\) are feasible for (P) and (D) then \(b^T y \leq c^T x\).

**Corollary 1.** Primal (dual) problem unbounded \(\Rightarrow\) dual (primal) problem infeasible.

**Corollary 2.** If \(x, y\) are feasible for primal and dual problem and \(c^T x = b^T y\) then \(x, y\) are optimal.

**Theorem.** (Strong duality.) (P) has an optimal solution \(x^*\) if and only if (D) has an optimal solution \(y^*\) and \(c^T x^* = b^T y^*\).
The simplex multipliers provides the dual solution at the optimum for the simplex algorithm:

\[ y = B^{-T}c_B \]
solves the dual problem if \( B \) is the basis of the optimal solution of the primal problem.

**Complementary slackness.** Let \( x^*, y^* \) solve \((P)\) and \((D)\). Strong duality:

\[ x^Tc = b^Ty^* = (Ax^*)^Ty^* = x^TA^Ty^*, \]

so

\[
0 = x^*(c - A^Ty^*) = \sum_{i=1}^{n} x_i^*(c - A^Ty^*)_i.
\]

\[ x_i^* > 0 \Rightarrow (c - A^Ty^*)_i = 0; \ (c - A^Ty^*)_i > 0 \Rightarrow x_i^* = 0 \]

Either \( x_i^* > 0 \) or \( (c - A^Ty^*)_i > 0 \) **not both!**
**Duality**

**Sensitivity:** The dual solution yields cost sensitivity for changes in \(b\)

\[
\min_{x} \zeta = c^T x \\
Ax = b \\
x \geq 0
\]

\(x^*\): nondegenerate optimal basic solution,

\[
x_B^* = B^{-1}b > 0 \\
x_N^* = 0
\]

Small changes \(\delta b\) in \(b\) does not change optimal basis so the change \(\delta x^*\) in \(x^*\) is

\[
\delta x_B^* = B^{-1} \delta b, \\
\delta x_N^* = 0
\]

and the change \(\delta \zeta\) in the objective \(z\) will be

\[
\delta \zeta = c^T \delta x^* = c_B^T \delta x_B^* + c_N^T \delta x_N^* = c_B^T B^{-1} \delta b = y^*^T \delta b
\]