Simplex Algorithm: Termination

**Theorem 2.14**

Assume that the simplex method is applied to an LP, and that each generated basic feasible solution is nondegenerate \((x_B > 0\) at each iteration). Then, in a finite number of iterations, the method finds an optimal solution or determines that the problem is unbounded.

For proof, see Theorem 5.10 in book.

**Degenerate case**: terminating with anti-cycling rule
**Example:**

- $n = 150$ variables, $m = 50$ constraints (a small problem!)
- Number of basic solutions, upper bound: $\binom{n}{m} \approx 10^{40}$ (many may be infeasible though)
- Assume: algorithms requires 1 ns per basis
- Takes $10^{23}$ years to examine all bases!
- $3m$ iterations take 150 ns

Artificial LPs can be constructed that force the simplex algorithm to examine every possible basis.

This behavior has not been observed for “real” problems (typically between $m$ and $3m$ iterations).
Simplex Algorithm: Initialization

Simplex method starts with a basic feasible solution (BFS). How to get this?

Easy for problems in inequality form!

\[
Ax \leq b \\
x \geq 0 \\
slacks \Rightarrow \\
Ax + s = b \\
x, s \geq 0
\]

\[
(A \ I) \begin{pmatrix} x \\ s \end{pmatrix} = b
\]

\[
\begin{pmatrix} x \\ s \end{pmatrix} \geq 0
\]

\[x_B = s, \ x_N = x = 0\] is a BFS if \(b \geq 0\)

For other problems, not so easy
Consider

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

with \( b \geq 0 \).

**Phase I:** Solve an *auxiliary* LP to obtain an initial basic feasible solution

**Phase II:** Solve the original LP
For phase I, introduce the *artificial variables* $y$ and consider

$$\min \sum_{i=1}^{m} y_i$$

subject to \hspace{1em} $Ax + y = b$

$x, y \geq 0$

an LP in $x, y$. Solve with simplex method. An initial BFS is $x = 0, y = b$ (since $b \geq 0$).

If there is a feasible solution to $Ax = b, x \geq 0$, (AP) has a solution with $\min \sum_{i=1}^{m} y_i = 0$. The simplex solution will be basic with respect to $A$. 
Solving (AP) to get an initial BFS usually works. Possible problems:

(a) If \( \min \sum_{i=1}^m y_i > 0 \), then \( Ax = b, x \geq 0 \) has no feasible solution!

(b) If \( \min \sum_{i=1}^m y_i = 0 \) and final solution of (AP) is degenerate, might happen that some \( y_i = 0 \) is in final basis. These basic variables can be exchanged with non-basic \( x_i \) variables (§ 5.5.1 in book)