The Coffee Grinder Problem

For some weird reason, a coffee grinder contains black and white coffee beans. We pick up two random beans. If the two beans have the same colour, we remove them from the grinder, and insert a new white bean. If the beans have different colours, we throw away the white bean, and put the black bean back in. If we repeat this procedure until there’s only one bean left, what is the colour of the last bean?

The program below implements this procedure. $b$ and $w$ denotes the number of black and white beans in the grinder, respectively.

\[
\begin{align*}
\{Q:\ b \geq 0 \land w \geq 0 \land b + w \geq 1 \land P'\} \\
\text{skip} \\
\{\text{inv } P:\ b \geq 0 \land w \geq 0 \land b + w \geq 1 \land P'\} \\
\{\text{bound } t:\ ?\} \\
\text{do} \\
\quad b \geq 2 \rightarrow b, w := b - 2, w + 1 \\
\quad w \geq 2 \rightarrow b, w := b, w - 1 \\
\quad b \geq 1 \land w \geq 1 \rightarrow b, w := b, w - 1 \\
\text{od} \\
\{R:\ ?\}
\end{align*}
\]
Assignment 1 (25%) Prove that our procedure terminates, ie find a bound function $t$, and prove it correct by proving clauses 4 and 5 of the iterative command theorem.

Assignment 2 (50%) Under which conditions is the last bean black? To answer this question, let $R = (b = 1 \land w = 0)$. Find a conjunct $P'$ of $P$ such that this post-condition holds and prove your choice correct by proving clauses 2 and 3 of the iterative command theorem.

Assignment 3 (25%) Under which conditions is the last bean white? Analogously, let $R = (b = 0 \land w = 1)$. Find a conjunct $P'$ of $P$ such that this post-condition holds and prove your choice correct by proving clauses 2 and 3 of the iterative command theorem.

For assignments 2 and 3, your choices of $P'$ must be non-trivial: silly choices such as $P' = F$ and $P' = R$ will not be considered.