Pricing of European Call Options

Scientific Computing Project Course

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Abstract

The pricing of a European call option on one underlying asset is carried out by means of two different methods: by numerically solving the parabolic partial differential equation known as the Black & Scholes' equation, and by using a Monte Carlo method, in which the price development of the asset is simulated and from that the value of the claim is computed as an expected value. The impacts of different volatilities, constant as well as stock price and time dependent, are investigated in the case of solving the Black & Scholes' equation. The PDE solver and the Monte Carlo method are compared regarding the efficiency in one and two dimensions. An option with two underlying assets is also briefly looked into using a Monte Carlo method.

Contents

1 Introduction

A call option is a derivative asset. It gives the holder of it the right (but not the obligation) to buy an asset, for example shares of a certain stock, which is called the underlying stock or the underlying asset. In this paper we focus entirely on European call options, which gives the holder the right to buy the stock on a specific date only, called the exercise date. The date of exercise is also called time of maturity and is often denoted by T. The price for which the asset may be bought on the exercise date is decided when the option is created, and this price is called the exercise price or the strike price, here denoted by K. Since this price cannot be altered, but naturally the value of the underlying asset can fluctuate, the value of the option is likely to change over time to adjust to the prevailing circumstances. If, on the date of exercise, the asset is worth more than the strike price, the holder will want to exercise the option, that is to use his or her right to buy the underlying asset. If he or she sells the stock in the stock market, then the gain will be the difference between the stock price and the exercise price. If on the other hand, the exercise price is higher than the stock price, the option holder is not likely to exercise the option. The call option is worthless.

The history of option trading is ancient. It is mentioned in the Bible (see [1]). It is known that early Phoenician traders dealt with options, as did the Greeks and the medieval merchants. The Dutch option market provided us with a famous (or notorious) financial crash. It took place in 1636; the centre of the economic catastrophe were tulips. Having become a status symbol of the aristocracy, the tulips were traded in a market where the prices spiralled out of control. The tulip speculations finally led to a serious damage of the Dutch economy, see [6]. Traditionally, farmers were often engaged in the options business, in order to secure payment for the crop before it was harvested. Although this kind of private option trading has been executed for centuries, it was not until the early 1900s when the American option market took a leap forward: Then the Put and Call Brokers and Dealers Association formed, and it brought some degree of structure to a formerly very unstructured market. Still grave weaknesses were present, and the market was not particularly stable. One of the drawbacks was that there was no universal guarantee, because the contracts were only backed by the selling brokerage. Option trading with stock shares as underlying assets commenced on Wall Street in the 1930s. During the course of the following decades, legislation and security improved, and a true breakthrough came in 1973, when the Chicago Board of Options Exchange opened. New rules and contract standardization made the market more accessible. Also clearing was introduced, so that the buyer and the seller could act independently. The very same year Fischer Black and Myron Scholes published their renown formula and their market model. Their work has been of significant importance and rendered stability to option markets worldwide. The Black & Scholes' model was in 1997 awarded the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel.

This report deals with the pricing of European call options in an environment with dynamics defined by the generalized Black & Scholes' model. Two different methods are used to compute the option price: a Monte Carlo method and an application of the Crank–Nicolson scheme to solve the Black & Scholes' partial differential equation. A comparative discussion of the two different methods is given.

2 Problem Presentation

2.1 The Black & Scholes' Market Model and Itô Calculus

In order to elucidate the basics of the option market, we have to assume an a priori given market with dynamics given by equations (1) and (2). Following conventions, we let B denote the price process of a risk free asset (bond), S the price process of a stock, r the interest rate and t the time. The interest rate is supposed to be constant over the observed period of time. The deterministic functions α and σ are both assumed to be known. They are functions of time and the price of the underlying asset. The function α is the local mean rate of return. This is the generalized Black & Scholes' market model.

$$dB(t) = rB(t)dt, \tag{1}$$
$$dS(t) = S(t)\alpha(t, S(t))dt + S(t)\sigma(t, S(t))dW(t). \tag{2}$$

The volatility of the underlying asset, σ, describes the tendency of the asset to change its value. The volatility can range from $\sigma = 0.1$ for a secure stock, to $\sigma = 0.8$ for a risky asset. W is a Wiener process, which is a stochastic process here used to simulate asset value developments. The formal definition of a Wiener process reads as follows:

- $W(0) = 0$.
- The process W has independent increments, i e if $s < u \leqslant v < w$, then $W(w) - W(v)$ and $W(u) - W(s)$ are independent stochastic variables.
- For $v < w$ the stochastic variable $W(w) - W(v)$ has the Gaussian distribution $\mathcal{N}(0, \sqrt{w - v})$.
- W has continuous trajectories.

The market is modelled by stochastic integral equations, and in order to handle the mathematics we have to exchange the ordinary calculus for Itô calculus. The major difference is that on taking differentials, we can no longer neglect all second order terms. We have to incorporate these rules:

$$\begin{cases} (dt)^2 = 0 \\ dtdW = 0 \\ (dW)^2 = dt \end{cases} \tag{3}$$

The main point is that $(dW)^2 \neq 0$. When differentiating a product the ordinary chain rule has to be complemented with a third term. Consider a function $f = f(t, X)$. Then we arrive at the well-known Itô Formula:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2. \tag{4}$$

Note the last term which makes the difference from the ordinary calculus chain rule.

2.2 The PDE Approach

By and large we now follow the line of discussion in [2].

Now that we can handle the needed calculus we return to the equations defining the market (1) and (2)

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dS(t) &= S(t)\alpha(t, S(t))dt + S(t)\sigma(t, S(t))dW(t). \end{aligned}$$

We consider a simple contingent claim of the form

$$X = \Phi(S(T)). \tag{5}$$

Φ is the contract function; we will return to this in a moment. We assume that this claim can be traded on the market and that its price process $\Pi(t) = \Pi(t; \Phi)$ has the form

$$\Pi(t) = F(t, S(t)), \tag{6}$$

for some smooth function F. Our problem is to find out what F must look like in order for the market $[S(t), B(t), \Pi(t)]$ to be free of arbitrage possibilities.

We start by computing the price dynamics of the derivative asset, and the Itô formula applied to (6) and (2) gives us (7).

$$d\Pi(t) = \alpha_\pi(t)\Pi(t)dt + \sigma_\pi(t)\Pi(t)dW(t), \tag{7}$$

where the process $\alpha_\pi(t)$ and $\sigma_\pi(t)$ are defined by

$$\alpha_\pi(t) = \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{F}, \tag{8}$$

$$\sigma_\pi(t) = \frac{\sigma S F_s}{F}. \tag{9}$$

Let us now form a portfolio based on two assets: the underlying stock and the derivative asset. Denoting the relative portfolio by (u_s, u_π) (relative here means how much each asset contributes to the total portfolio value) and using the fact that it is a self-financing portfolio, i e there is no withdrawal or infusion of money, we obtain the following dynamics for the value V of the portfolio

$$dV = V\left\{u_s\left[\alpha dt + \sigma dW\right] + u_\pi\left[\alpha_\pi dt + \sigma_\pi dW\right]\right\}. \tag{10}$$

We now collect dt- and dW-terms to obtain

$$dV = V\left[u_s\alpha + u_\pi\alpha_\pi\right]dt + V\left[u_s\sigma + u_\pi\sigma_\pi\right]dW. \tag{11}$$

The point to notice here is that both brackets are linear in the arguments u_s and u_π. Let us thus define the relative portfolio by the linear system of equations

$$u_s + u_\pi = 1, \tag{12}$$

$$u_s\sigma + u_\pi\sigma_\pi = 0. \tag{13}$$

Equation (12) comes from that fact that it is a relative portfolio. Using this portfolio we see that by its very definition the driving dW-term in the V-dynamics of (11) vanishes completely, leaving us with the equation

$$dV = V[u_s\alpha + u_\pi\alpha_\pi]dt. \tag{14}$$

Thus we have obtained a locally riskless portfolio, and because of the requirement that the market is free of arbitrage we see from equation (14) that the following must hold

$$u_s\alpha + u_\pi\alpha_\pi = r. \tag{15}$$

This is thus the condition for absence of arbitrage, and we will now look more closely at this equation.

It is easily seen that the system (12),(13) has the solution

$$u_s = \frac{\sigma_\pi}{\sigma_\pi - \sigma}, \tag{16}$$

$$u_\pi = \frac{-\sigma}{\sigma_\pi - \sigma}, \tag{17}$$

which, using (9), gives us the portfolio more explicitly as

$$u_s(t) = \frac{S(t)F_s(t,S(t))}{S(t)F_s(t,S(t)) - F(t,S(t))}, \tag{18}$$

$$u_\pi(t) = \frac{-F_s(t,S(t))}{S(t)F_s(t,S(t)) - F(t,S(t))}. \tag{19}$$

Now we substitute (8), (18) and (19) into the absence of arbitrage condition (15). Then, after some calculations, we obtain our result in the equation (20) describing the price process of the claim:

$$F_t(t,S(t)) + rS(t)F_s(t,S(t)) + \frac{1}{2}S(t)^2\sigma^2F_{ss}(t,S(t)) - rF(t,S(t)) = 0. \tag{20}$$

For every fixed $t \in (0,\infty)$, the distribution of $S(T)$ has support over the entire real line, and thus $S(T)$ in (20) can be exchanged for s, a variable of its own. Adding a terminal condition (see below) we obtain the Black & Scholes' equation:

$$F_t(t,s) + rsF_s(t,s) + \frac{1}{2}s^2\sigma^2F_{ss}(t,s) - rF(t,s) = 0, \tag{21}$$

$$F(T,s) = \Phi(s). \tag{22}$$

Here $\Phi(s)$ is the so called contract function, the terminal condition giving the value of the option at the date of maturity, denoted T. The terminal condition for a European call option is $\max(S(T) - K, 0)$, where K is the strike price of the option.

The PDE (21), (22) is a diffusion (heat) equation with a terminal condition instead of, as in most physical systems, an initial condition. (The time is in this PDE reversed compared to the ordinary physical diffusion equation.) In the case of a non-constant σ, there is no analytical expression for the price function F (except in the case of a stock price dependent volatility of the form $s^{-\alpha}$, but these expressions are generally very complicated). Therefore the aim is to find a numerical solution of (21), (22). With constant volatility, equation (2), describing the dynamics of the stock price, is a Geometric Brownian Motion. An explicit solution for this case is the celebrated Black & Scholes' formula:

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)], \tag{23}$$

where $F(t,s) = \Pi(t)$, N is the cumulative distribution function for the $\mathcal{N}(0,1)$ distribution and

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\frac{s}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \tag{24}$$

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t}. \tag{25}$$

For details of this section we refer to [3] and [2].

2.3 The Stochastic Representation Approach

The stochastic approach is based upon the fact that the option price process has a so called Feynman–Kač stochastic representation, see the following derivation.

Consider an environment with dynamics given by the generalized Black & Scholes' model (1), (2), (where $\alpha = r$) and a claim of the form $X = \Phi(S(T))$ on an underlying asset satisfying

$$dS(t) = rS(t)dt + S(t)\sigma(t,S(t))d\bar{W}(t). \tag{26}$$

We now have a new probability distribution, compared to the probability distribution of (2). This new distribution is indicated by the letter Q. $\bar{W}$ is a wiener process under this new measure, whereas w was under the former measure. The price process of the claim is denoted by F. It satisfies the Black & Scholes' equation (21), (22) for every fixed positive value of t. Since $\operatorname{supp}(S(t)) = \mathbb{R}$, (21), (22) can be replaced by

$$F_t(t,S(t)) + rS(t)F_s(t,S(t)) + \frac{1}{2}S(t)^2\sigma^2 F_{ss}(t,S(t)) - rF(t,S(t)) = 0, \tag{27}$$

$$F(T,S(T)) = \Phi(S(T)), \tag{28}$$

see [2].

The Itô Formula (4) applied on the price process F gives

$$\begin{aligned} dF(t,S(t)) &= F_t dt + F_s dS(t) + \frac{1}{2}F_{ss}(dS(t))^2 \\ &= F_t dt + F_s[rS(t)dt + S(t)\sigma(t,S(t))d\bar{W}(t)] + \frac{1}{2}F_{ss}((S(t))^2(\sigma(t,S(t)))^2 dt) \\ &= (F_t + rS(t)F_s + \frac{1}{2}S(t)^2\sigma^2 F_{ss})dt + F_s S(t)\sigma(t,S(t))d\bar{W}(t) \\ &= rF(t,S(t))dt + F_s S(t)\sigma(t,S(t))d\bar{W}(t). \end{aligned}$$

This implies

$$\begin{aligned}\int_t^T dF(\tilde{t}, S(\tilde{t})) &= \int_t^T rF(\tilde{t}, S(\tilde{t}))d\tilde{t} + \int_t^T F_s S(\tilde{t})\sigma(\tilde{t}, S(\tilde{t}))d\bar{W}(\tilde{t}) \\ &= F(T, S(T)) - F(t, S(t)).\end{aligned}$$

With $F_s S(\tilde{t})\sigma(\tilde{t}, S(\tilde{t})) \in \pounds^2$, the expected value of the stochastic integral equals zero (see [2]) and we get

$$\begin{aligned}\int_t^T rE^Q[F(\tilde{t}, S(\tilde{t}))]d\tilde{t} &= E^Q[F(T, S(T)) - F(t, S(t))] \\ E^Q[F(t, S(t))] &= E^Q[F(T, S(T))] - \int_t^T rE^Q[F(\tilde{t}, S(\tilde{t}))]d\tilde{t}.\end{aligned}$$

Taking time derivatives yields

$$\begin{aligned}E^Q[F(\dot{t}, S(t))] &= rE^Q[F(t, S(t))] \\ E^Q[F(t, S(t))] &= Ce^{rt} \\ C &= e^{-rt}E^Q[F(t, S(t))].\end{aligned}$$

Furthermore

$$E^Q[F(t, S(t))]|_{t=T} = e^{-rt}E^Q[F(t, S(t))]e^{rT} = e^{r(T-t)}E^Q[F(t, S(t))],$$

and since the expected value of a deterministic function is the observed value,

$$E^Q[\Phi(S(T))] = e^{r(T-t)}F(t, S(t)).$$

Thus the arbitrage free price process of a claim of the form $X = \Phi(S(T))$ has a Feynman–Kač stochastic representation:

$$F(t, S(t)) = e^{-r(T-t)}E^Q[\Phi(S(T))]. \tag{29}$$

This is the background to the straight-forward Monte Carlo approach.

3 Solving the Black & Scholes' Equation

3.1 Setup

3.1.1 The Scheme

Inspired by Vázques, [12], the Crank–Nicolson scheme was used to solve (21 on page 6) and (22 on page 6). We improved the approximation of the first spatial derivative to be a centered approximation in both time steps. The interest rate term was handled analogously. The Crank–Nicolson scheme is accurate of order (2,2) and due to the intrinsics of the PDE, unconditionally stable. The Crank–Nicolson scheme reads as follows, with time discretization step k and stock price step h:

$$\frac{v_m^n - v_m^{n+1}}{k} + \frac{rs_m}{2}\left(\frac{v_{m+1}^n - v_{m-1}^n + v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h}\right) +$$
$$\frac{\sigma^2 s_m^2}{4}\left(\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n + v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}}{h^2}\right) - r\left(\frac{v_m^n + v_m^{n+1}}{2}\right) \quad = \quad 0.$$

This is equivalent to

$$v_{m-1}^{n+1}\left(\frac{rs_m}{4h} - \frac{1}{2}\frac{\sigma^2 s_m^2}{2h^2}\right) + v_m^{n+1}\left(\frac{1}{k} + \frac{\sigma^y 2 s_m^2}{2h^2} + \frac{r}{2}\right) + v_{m+1}^{n+1}\left(-\frac{1}{2}\frac{\sigma^2 s_m^2}{2h^2} - \frac{rs_m}{4h}\right) =$$
$$v_{m-1}^n\left(\frac{1}{2}\frac{\sigma^2 s_m^2}{2h^2} - \frac{rs_m}{4h}\right) + v_m^n\left(\frac{1}{k} - \frac{\sigma^2 s_m^2}{2h^2} - \frac{r}{2}\right) + v_{m+1}^n\left(\frac{1}{2}\frac{\sigma^2 s_m^2}{2h^2} + \frac{rs_m}{4h}\right),$$

equvivalent to

$$v_{m-1}^{n+1}\left(\frac{rs_m\lambda}{4} - \frac{1}{2}\frac{\sigma^2 s_m^2\mu}{2}\right) + v_m^{n+1}\left(1 + \frac{\sigma^2 s_m^2\mu}{2} + \frac{rk}{2}\right) + v_{m+1}^{n+1}\left(-\frac{1}{2}\frac{\sigma^2 s_m^2\mu}{2} - \frac{rs_m\lambda}{4}\right) =$$
$$v_{m-1}^n\left(\frac{1}{2}\frac{\sigma^2 s_m^2\mu}{2} - \frac{rs_m\lambda}{4}\right) + v_m^n\left(1 - \frac{\sigma^2 s_m^2\mu}{2} - \frac{rk}{2}\right) + v_{m+1}^n\left(\frac{1}{2}\frac{\sigma^2 s_m^2\mu}{2} + \frac{rs_m\lambda}{4}\right),$$

where as usual $\lambda = \frac{k}{h}$ and $\mu = \frac{k}{h^2}$.

3.1.2 The Parameters

Throughout the tests, the interest rate r is set to four percent. The strike price is $K = 30$, and the exercise date T is one year from today.

We used different volatilities σ for the underlying asset: constant σ (for reference solutions, see below), σ depending on the stock price only ($\sigma \propto s^{-\alpha}$, $\alpha \in [0,1]$) and different volatilities independent of the stock price but changing over time. In the first tests we modelled volatilities with $\alpha = 0.4$ and $\alpha = 0.8$. These are our time dependent volatilities:

Ascending staircase volatility

$$\sigma = \begin{cases} 0.1 & t \in (\text{today}, \frac{T}{4}] \\ 0.2 & t \in (\frac{T}{4}, \frac{T}{2}] \\ 0.4 & t \in (\frac{T}{2}, \frac{3T}{4}] \\ 0.8 & t \in [\frac{3T}{4}, T] \end{cases}$$

Descending staircase volatility

$$\sigma = \begin{cases} 0.8 & t \in (\text{today}, \frac{T}{4}] \\ 0.4 & t \in (\frac{T}{4}, \frac{T}{2}] \\ 0.2 & t \in (\frac{T}{2}, \frac{3T}{4}] \\ 0.1 & t \in [\frac{3T}{4}, T] \end{cases}$$

Crash volatility

$$\sigma = \begin{cases} 0.2 & t \in (\text{today}, \frac{T}{4}) \\ e^{-t} & t \in [\frac{T}{4}, T] \end{cases}$$

The Ascending staircase volatility function describes a stock becoming more unstable, whereas the Descending staircase volatility function describes a stock calming down. The Crash models a stock with a low volatility to begin with, $\sigma = 0.2$, and three months from now (the exercise date being one year ahead) the volatility rockets to a high level, thereafter recovering comparatively swiftly.

3.1.3 The Boundary Conditions

The boundary conditions are chosen to comply with natural restrictions. At $s = 0$, the boundary values are simply set to zero, because the option price cannot take on a positive value unless the stock itself is worth something. The upper boundary $s_{\max}$ was set at approximately 3 times the strike price (a common rule of thumb, see [7]). The values on the upper boundary are taken from [12]: the maximum stock price minus the discounted strike price, $F(t, s_{\max}) = s_{\max} - Ke^{-r(T-t)}$.

As terminal condition we use the contract function for a European call option, $\max(S(T) - K,\ 0)$.

3.2 Results

Results of our pricing model are presented graphically. Note that all results have been computed for the entire interval $x \in [0, 100]$, but in the result figures 1, 3, 5 and 7 we printed only the interval $x \in [10, 50]$, for the sake of clarity. (The region of interest is within the chosen interval because the strike price is $K = 30$.)

The first graph (figure 1) shows option prices for two different stock price dependent volatilities $\sigma = S^{-\alpha}$ with $\alpha = 0.4$ and $\alpha = 0.8$ respectively. Figure 2 shows the volatilities corresponding to the results of figure 1. The following three results (figures 3, 5 and 7) depict option prices where the volatilities (figures 4, 6 and 8) depend on time only.

In the graphs with time dependent volatilities, we included reference solutions to compare the results to: The reference solutions represent the outcome had the volatility not been time dependent, but stayed constant at the initial value all over the considered period of time. (The constant volatility case gives the usual Black & Scholes' model. The solutions to these cases were given by the Black & Scholes' formula equation(23).) The time span is in all cases one year (the time axis in the graphs depicting the volatilities are graded in years).

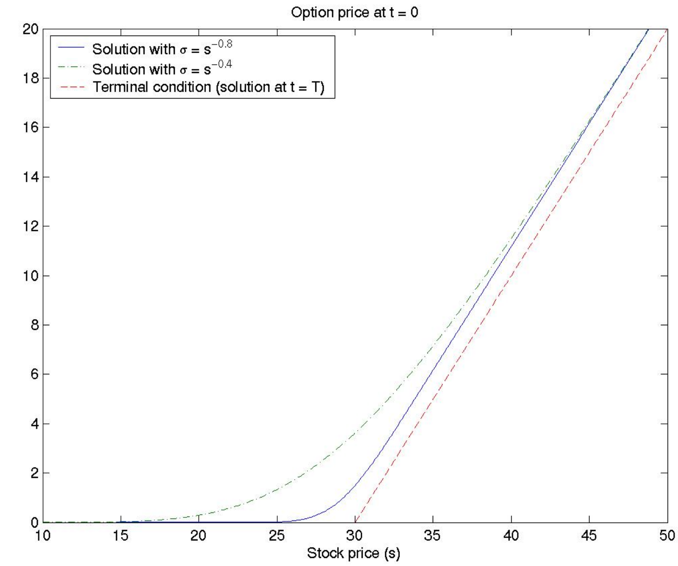

Figure 1: Option price for α=0.4 and α=0.8.

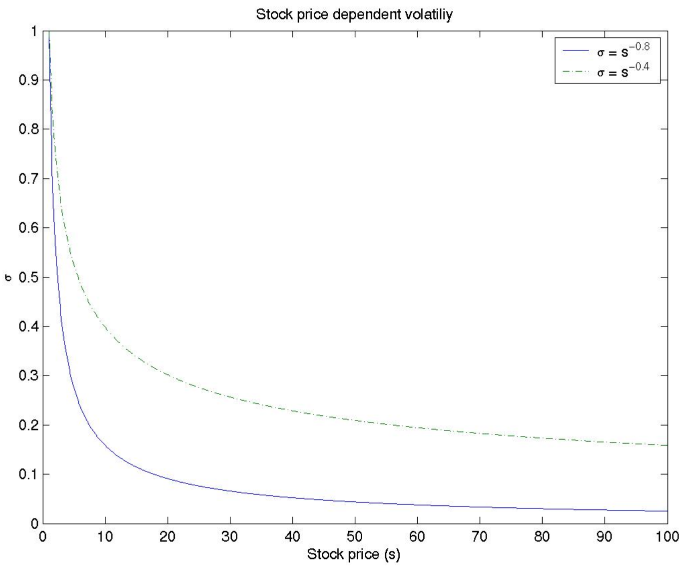

Figure 2: Volatility for α=0.4 and α=0.8.

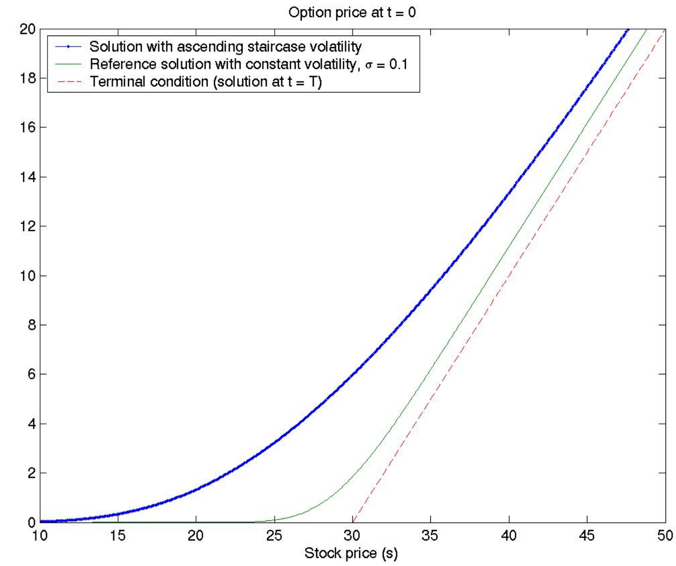

Figure 3: Option price for Ascending staircase volatility function.

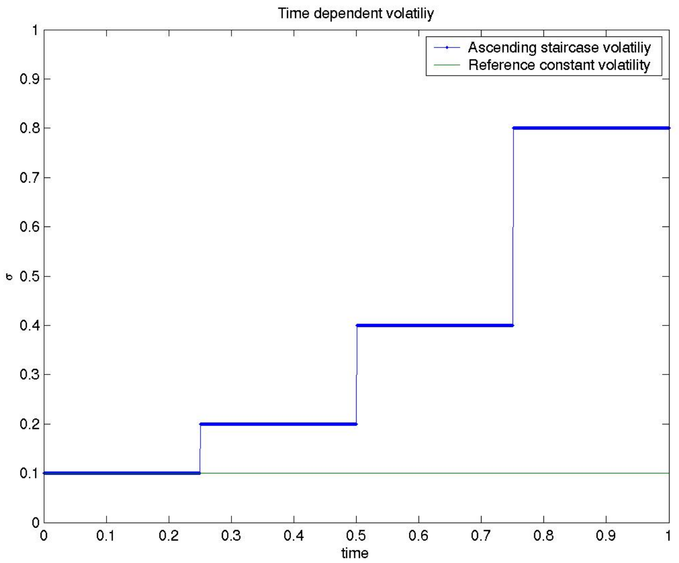

Figure 4: Ascending staircase volatility function.

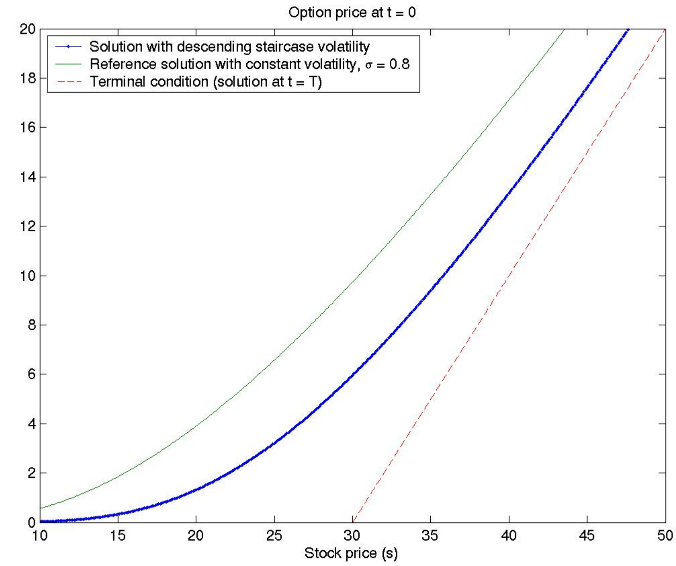

Figure 5: Option price for Descending staircase volatility function.

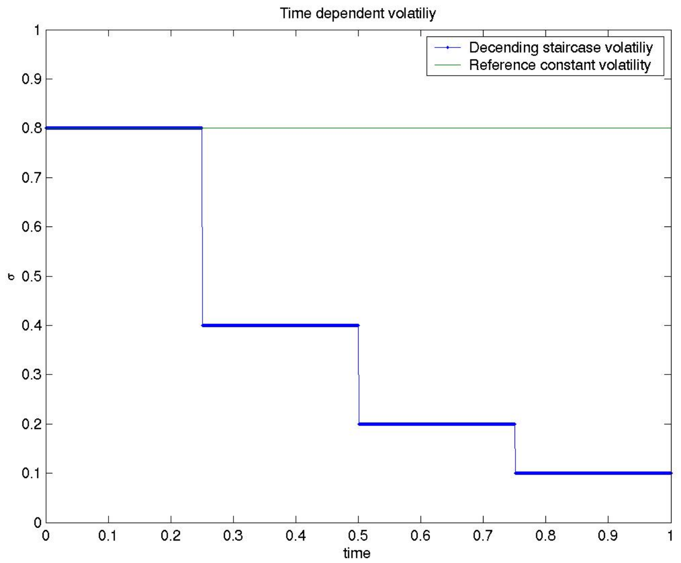

Figure 6: Descending staircase volatility function.

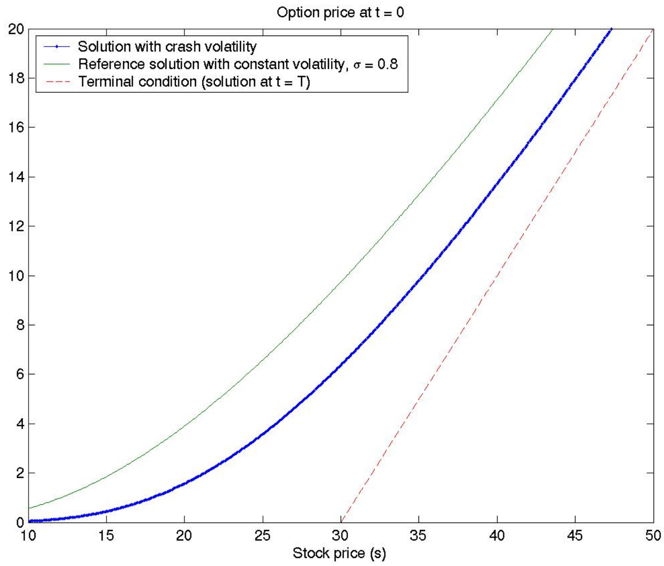

Figure 7: Option price for Crash volatility function.

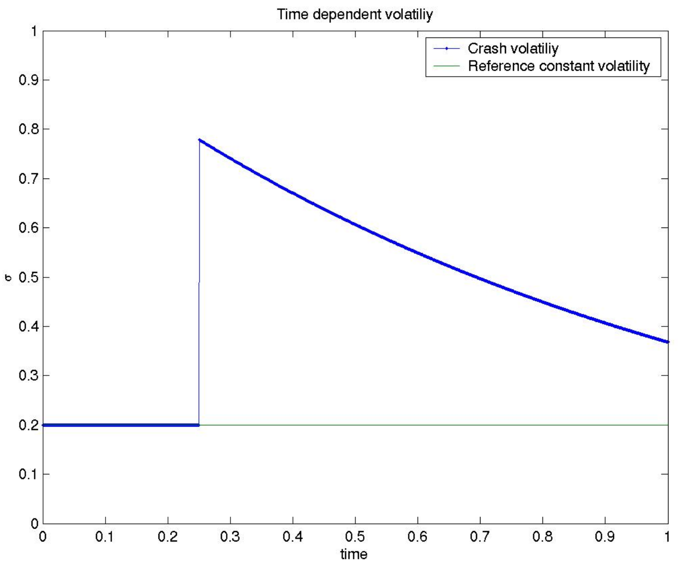

Figure 8: Crash volatility function.

The first figure shows that despite a small volatility (the $\sigma = s^{-0.8}$ case), there is a gap in between the option price solution and the terminal condition due to the interest rate. This can also be seen in the other graphs. Were there no interest rate, the solutions would stick closer to the terminal condition. All the results confirm a known relation: the larger the volatility, the higher the option price.

3.3 Error Analysis

3.3.1 The Accuracy

The order of accuracy of the Crank–Nicolson scheme is as mentioned (2,2). This can be verified by means of forming Romberg control quotients, $\frac{v_h^k - v_{h/2}^k}{v_{h/2}^k - v_{h/4}^k}$ and $\frac{v_h^k - v_h^{k/2}}{v_h^{k/2} - v_h^{k/4}}$. The quotients are displayed in Table 1.

Constant time step: k = 10^{-3}	h = 0.01	h = 0.004
Control Quotients	3.997	4.001
Log_2(Control Quotients)	1.999	2.000

Constant space step: h = $5 * 10^{-3}$	k = $5 * 10^{-3}$	k = $1.25 * 10^{-3}$
Control Quotients	4.000	4.000
Log_2(Control Quotients)	2.000	2.000

Table 1: Romberg control quotients.

3.3.2 The Boundary Condition

In the solving of the Black & Scholes' equation, the Crank–Nicolson scheme requires a boundary condition on the upper boundary $s_{\max}$ of the stock price dimension. The price function value is there set to be $F(t, s_{\max}) = s_{\max} - Ke^{-r(T-t)}$. This is correct only if the contract function is $S(T) - K$, and not, as under these circumstances, $\max(S(T) - K,\ 0)$. However, the discrepancy between these two functions is rather small if $s_{\max} >> K$; it can be shown that what can be regarded as the true solution, is the upper limit of the solution acquired with the boundary condition used here. Also, since the error decreases as $s \to \infty$, the upper stock price limit should be set relatively high. This supports the common rule of thumb of cutting the stock price axis at three or four times the exercise price.

4 The Monte Carlo Approach

4.1 The Monte Carlo Method

4.1.1 The Monte Carlo Models

The Feynman–Kač stochastic representation (29 on page 8), allows us to decide the option price when knowing the expected value of the contract function. This value is simulated by taking the average of a large number of simulations, each representing a possible price development of the stock. (A single result of the Monte Carlo method may or may not be representative for the common case. To obtain reliable results a great many simulations need to be executed. This makes the Monte Carlo method inefficient, as we see in the sequel.)

Two Monte Carlo methods were implemented, both can be found in [8]. The first one is an implementation of the exponential expression for the price of the underlying asset itself:

$$S_t = S_{t-1} e^{((r - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W)}, \tag{30}$$

where ΔW describes the independent increments of the Wiener process and $\Delta W \in \mathcal{N}(0, \sqrt{\Delta t})$. This model of the asset price is exactly correct when the volatility σ is piecewise constant. When the volatility is continuously differentiable, the model becomes an approximation of the true solution. The implementation of (30) was used in all Monte Carlo computations except when deciding the time discretization error. There are two reasons for this. When a constant volatility is considered, only one time step is needed, and so the time discretization error is fully eliminated. (A single time step suffices because each Wiener process increment is independent of the others, and we know the distribution of the final step.) On the other hand, when a non-constant volatility is considered, no analytical solution exists, and so it is impossible to get an analytical expression for the error anyhow.

The second implementation is a linear approximation of (30), (according to [8] not recommended for actual use, but still widely used, see e g [5]):

$$S_t = S_{t-1} + r\Delta t S_{t-1} + \sigma S_{t-1}\Delta W. \tag{31}$$

This model is used when deciding the time discretization error.

4.1.2 Monte Carlo Properties

The Monte Carlo method does not yield solutions for all possible stock prices, like the Crank–Nicolson method does. Based on one current stock price and the exercise price each simulation naturally gives one option price. In a Monte Carlo test, a large number of simulations are carried out simultaneously: starting from today's asset value, random changes are added, one for each time step in the time discretization. When having reached the exercise date, possible future values of the asset are obtained, and from that the corresponding option prices are computed according to the terminal condition (in this case $\max(S(T) - K, 0)$). Finally, a single mean value of the results is computed and discounted to today's date. The result is therefore not very well suited for graphs unless one solves for many different stock prices. An advantage of the Monte Carlo method is precisely this: it is more natural to obtain a single option value, because it corresponds to the actual situation. The underlying option has only one value at a time, and to get superfluous information about option values in case the asset value would had been different, is not interesting.

4.2 Results

The solutions obtained from the Monte Carlo method is of course just as correct as the PDE solution, only the accuracy is not as high. Figure 9 on the following page depicts results of the Monte Carlo method together with the Crank–Nicolson solution from figure 3 on page 12. As can be seen, the different solutions are quite comparable.

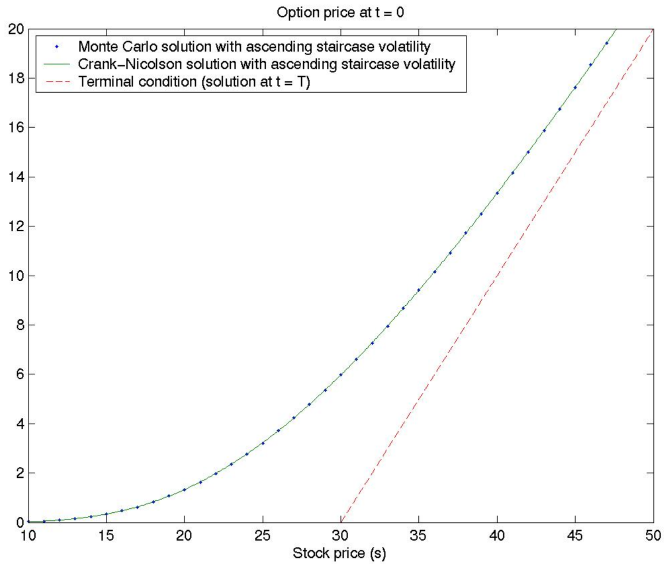

Figure 9: Option prices obtained with the Monte Carlo method for Ascending staircase volatility function.

With several underlying assets the Monte Carlo method becomes more interesting since this is very easily implemented, you simply simulate more assets. There can also be dependencies between the assets, for example stocks within the same market area.

Below is an example of a so called exchange option which gives the holder the right to trade a stock for another. In this case the contract function becomes $\max(S_1(T) - S_2(T), 0)$, where S_2 is the stock held and S_1 the stock to change to. For two independent assets with constant volatility the price is given by a generalization of the Black & Scholes' formula (23 on page 7). In figure 10 this formula is used and we see that the Monte Carlo model is accurate. In figure 11 we have two dependent assets which generates extra volatilities for the assets so that they become dependent on the Wiener-process of the other asset as follows

$$dS_1 = rS_1dt + \sigma_{11}dW_1 + \sigma_{12}dW_2,$$
$$dS_2 = rS_2dt + \sigma_{21}dW_1 + \sigma_{22}dW_2.$$

Such a case is depicted in figure 11, (the Monte Carlo method was used to compute the solution), with volatilities $\sigma_{11} = 0.4$, $\sigma_{12} = 0.1$, $\sigma_{21} = 0.05$ and $\sigma_{22} = 0.3$. The interest rate is set to zero. The reference solution is the solution for independent assets ($\sigma_{12} = 0$ and $\sigma_{21} = 0$), where we could use the analytical expression.

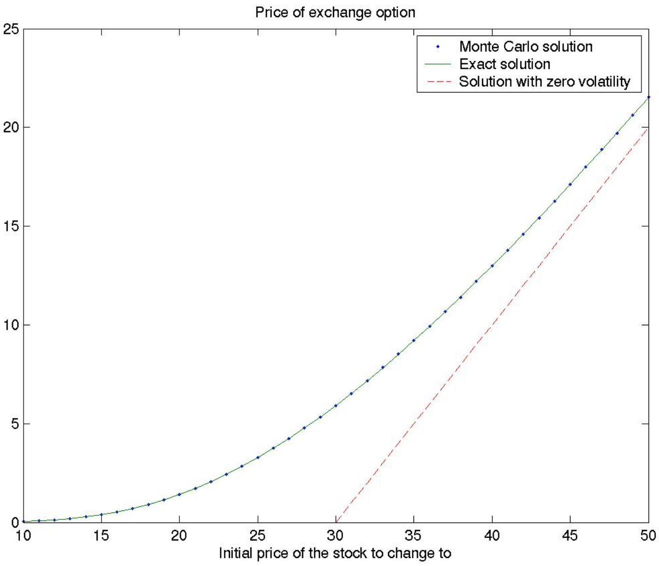

Figure 10: Price of an exchange option on two independent underlying assets.

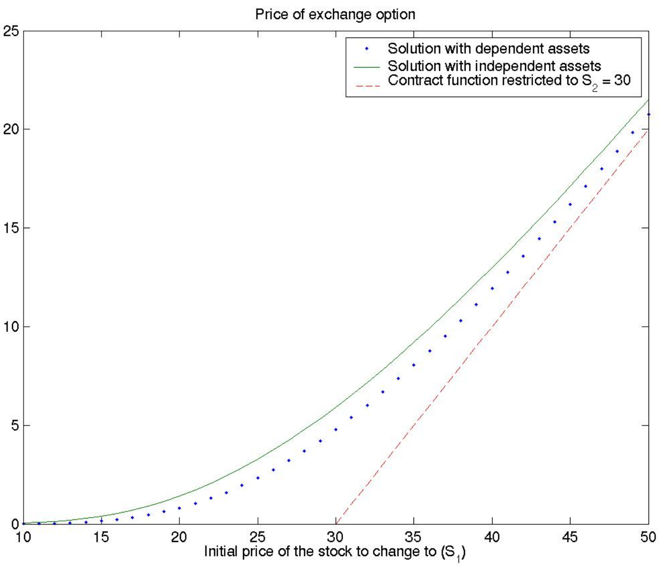

Figure 11: Comparison of the price of exchange options on two independent and depentent assets respectively.

In figure 11 we see that despite the extra volatility added to the dependent assets the price of the exchange option is less than that of the exchange option with independent assets. This can be interpreted as the dependent assets tend to rise and fall together which makes them more stable relative to one another.

4.3 Error Analysis

The Monte Carlo method gives rise to two errors: one statistical error and one originating from the discretization in time.

4.3.1 The Statistical Error

The statistical error for a general Monte Carlo method is proportional to $\frac{1}{\sqrt{n}}$, where n is the number of simulations, see [4] and [11]. In our tests to pin down the statistical error, we used the implementation of (30), because only one time step is needed, and so the time discretization error is totally eliminated. The following graph illustrates the convergence for our exponential Monte Carlo method: a simulation with altogether one hundred million paths, where the mean value of the error so far is plotted every millionth path. Even though it can be hard to discover a pattern in the successive errors, in the graph we can discern some sort of convergence. This behaviour is a randomness characteristic.

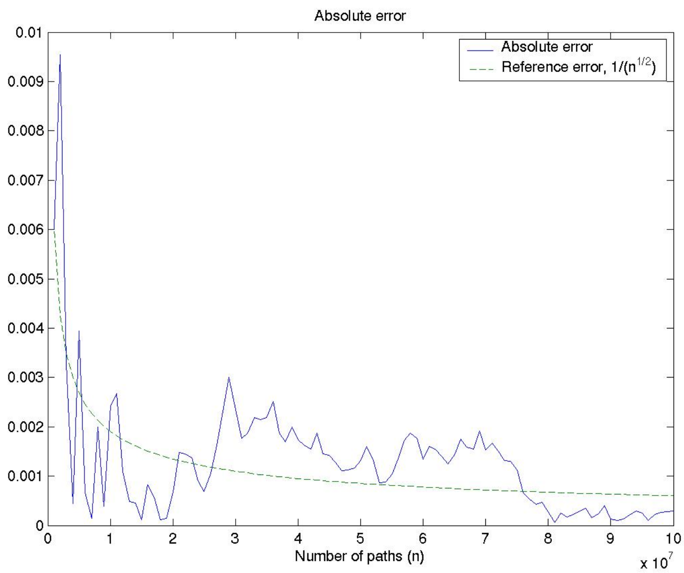

Figure 12: Convergence of a Monte Carlo method based on (30 on page 15).

4.3.2 The Discretization Error

The time discretization error decreases proportionally to the inverse of the number of time intervals m, see [11]. The following graph, resulting from a test of the implementaion of (31), illustrates this decrease. It is a graph of the total error but with a relatively small statistical error so that the time discretization error can be seen. Without the statistical error the graph would be smooth. Still the result is relatively close to the reference error $\frac{1}{m}$.

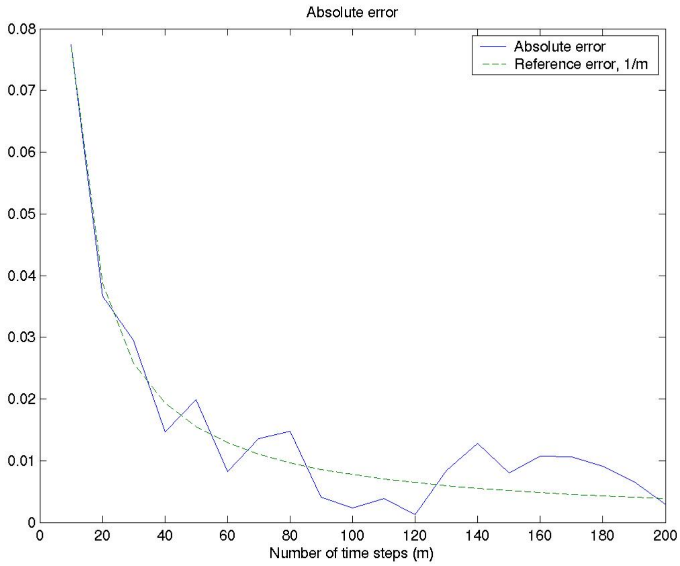

Figure 13: Absolute error for a Monte Carlo method.

5 Comparison

5.1 Efficiency

A comparison of the efficiency of the two methods shows that the Crank–Nicolson scheme is to be preferred in one dimension. The table below presents the number of arithmetical operations required for specified accuracies (errors). For the Crank–Nicolson scheme the number of steps in the time discretization equals the number of steps in the stock price discretization. The errors for the Monte Carlo method presented in the table are purely statistical.

Number of Arithmetical Operations

Required Accuracy:	error $= 10^{-3}$	error $= 0.5 * 10^{-3}$
Crank–Nicolson scheme	$6.5 * 10^5$	$1.2 * 10^6$
Monte Carlo method	$4.0 * 10^8$	$1.6 * 10^9$

Table 2: Comparison of Efficiency.

5.2 Multi-dimensional Computations

The Monte Carlo method has its obvious disadvantages, e g to decide the order of accuracy is a rather difficult task, since patterns of behaviour may be hard to get at, even for very large numbers of simulations. This difficulty arises from the Monte Carlo property of being based on randomness. Precisely this unstructured quality also forms the strength of the method, because it makes possible what in context of other methods would be very complicated, namely solving multi-dimensional problems. The Crank–Nicolson scheme does not present any problems in the respect of deciding the accuracy, but it has disadvantages emerging in the multi-dimensional case.

A one-dimensional problem is based on one single underlying stock only, whereas a multi-dimensional problem deals with a number of underlying assets. When generalizing the Crank–Nicolson scheme to higher dimensions, the matrix to be inverted in every time step is much more complicated to invert than the tridiagonal matrices of the one-dimensional case. The increase of arithmetical operations and execution time makes the multi-dimensional Crank–Nicolson unmanagable. (In addition to the excessive increase of arithmetical operations the scheme itself is also complicated to implement.) Compare the one-dimensional Crank–Nicolson scheme to the two-dimensional Crank–Nicolson scheme for solving (27), (28): The coefficient matrix is a band matrix of size $(N_x N_y) \times (N_x N_y)$ with a bandwith of $2N_x$. In our implementation we use LU factorization in the Crank–Nicolson algorithm, and for a $m \times m$ matrix with a bandwith of w the LU factorization can be carried out in $O(mw^2)$ operations. Substitutions to solve the system of equations require $O(mw)$ operations in each time step. Thus, for a solution with N_t time steps, in the one-dimensional case with constant volatility, where we have no discretization in the y direction, the overall number of operations needed is $O(N_x) + N_t O(N_x)$. For the 2D case the number is $O(N_x^3 N_y) + N_t O(N_x^2 N_y)$. Therefore the Crank–Nicolson scheme is not recommended for multidimensional computations and one should in stead consider more efficient methods such as the Alternating Direction Implicit (ADI) method, see further [10] and [9]. Monte Carlo methods can easily be implemented for multidimensional computations, and therein lies an important advantage of the method.

6 Discussion

The Black & Scholes' generalized market model (1) and (2) can be questioned. But so far no one has come up with something better, and so this model is considered to satisfactorily model option markets. A difficulty attached to the model is that the volatility is unknown. In fact the volatility is often computed by applying Black & Scholes' formula (23) to today's option market prices. The volatility obtained in this way is called implied volatility. Another way to get information about the volatility is to use historical data of the price development of the underlying asset. Neither of these methods yields a volatility that could be regarded as "true" – a "true" volatility does not even exist due to the unpredictable nature of the stock exchange. To improve the model and the volatility predictions is a task occupying researchers all over the world.

Given (21) and (22) with one underlying asset only, a PDE solver would give a reliable result. The equation, being a diffusion equation, is well known and presents no significant solution difficulties.

Monte Carlo methods are so imprecise so they should only be used when all alternatives are worse. However, they can provide a simple overview of the situation.

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