Current distribution in conductors

by

Lars Hägglund och Johan Sandström

Internrapport nr. 2003:5

UPPSALA UNIVERSITET
Inst. för informationsteknologi
Avd. för teknisk databehandling

UPPSALA UNIVERSITY
Information Technology
Dept. of Scientific Computing

Abstract

This report will explain what current distribution in conductors is, and how to calculate the effect. It will go into some depth on the theoretical background and describe an example of solving the problem numerically. It also leads to a conclusion that in some cases the current and thermal distributions can be viewed as uniform. The report will *not* go into the physical processes that leads to the current distribution.

Contents

1 Introduction

The term "current distribution" is used throughout this report to indicate the phenomenon that a current does not always distribute itself uniformly in a conductor. The purpose of this report is to give a tool for calculating the current distribution. This effect is visible during short current pulses (transients) or in the first microseconds when a DC current is turned on. The nonuniform distribution is also present in a high frequency harmonic current. The property that the current becomes nonuniformly distributed, when the input current is a harmonic current, will later be useful for both analytical and numerical reasons.

It also turns out that the uneven distribution of current means that the temperature will be nonuniformly distributed, as the current gives a heating effect. So to calculate the distribution of the current in the conductor there will also be a need to calculate the temperature distribution in the conductor.

A general analytical solution of how the current is distributed in the conductor is not known, and for most inputs the resulting distributions can not be analytically calculated. This means that the distribution must be calculated numerically. The same conclusion is also valid for the temperature, as it is directly dependent on the current distribution.

1.1 What assumptions can be made about the conductor

The first assumption made about the conductor is that it is cylinder shaped with radius a and of length l. The second assumption is that the length l is large enough to approximate to infinity. These are reasonable assumptions for most normal conductors, as the most common conductor is a cylinder shaped metal wire. These assumptions also gives a possibility to simplify the analytical part of the problem very much before numerical methods must be applied.

1.2 What parameters will the distribution depend on?

The distribution of the current is, as stated before, dependent on the temperature of the conductor. The distribution is also dependent on the radius of the conductor and the material properties of the conductor. Finally the current distribution is dependent on the size and form of the input current.

As the current is flowing through the conductor, the thermal effect will make the conductor heat up. The heating effect will be distributed similarly to the current. The temperature then gives differences in electrical conductivity and thus further differences in the current distribution.

1.3 Is the temperature distribution relevant ?

The question of the temperature distribution's relevancy comes from the fact that the material that is primarily studied, copper, has a very good thermal conductivity. The thermal conductivity might actually be so good that the temperature can be approximated to be uniformly distributed inside the conductor. An approximation such as that would simplify the calculations a lot, but it would only be relevant for copper and some other materials with high thermal conductivity, and not for any other material.

So, for some materials, the temperature distribution in the conductor will be such that it *must* be taken in account, the speed with which the heat spreads in these is too low to be neglected.

2 Theoretical analysis

In this section the analytical part of the different sub-problems will be taken as far as possible. It will also have a brief discussion on the numerical implications of the formulas in the end of each subsection. The section is split into three different subsections. The first subsection is only about the current distribution. The second subsection is about thermal transport in a conductor. The third subsection is about the coupling between the current distribution and the thermal distribution.

2.1 Current distribution

One of the first things that is needed is to theoretically examine the distribution of the magnetic field inside the conductor. This is because the magnetic field is tightly coupled to the current. The following are definitions of the symbols that will be used in this section.

$\boldsymbol{H}$	$[A/m]$	Magnetic field strength (magnetic intensity)
$\boldsymbol{E}$	$[V/m]$	Electric field strength (electric intensity)
$\boldsymbol{B}$	$[T; Wb/m^2; Vs/m^2]$	Magnetic flux density (magnetic induction)
$\boldsymbol{D}$	$[C/m^2; As/m^2]$	Electric flux density (electric displacement)
$\boldsymbol{j}$	$[A/m^2]$	Current density
σ	$[A/Vm]$	Conductivity
μ	$[Vs/Am]$	Permeability
ε	$[As/Vm]$	Permittivity

Following by one of Maxwell's equations, the current density can, via the magnetic field, be calculated. This is according to the formula

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{1}$$

This together with Ohm's law $\boldsymbol{j} = \sigma \boldsymbol{E}$ and the relation $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ gives

$$\nabla \times (\nabla \times \boldsymbol{H}) = \sigma(\nabla \times \boldsymbol{E}) + \varepsilon \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E}). \tag{2}$$

The right hand side of this equation is, via the vector identity and Maxwell's equation $\nabla \cdot \boldsymbol{B} = 0$,

$$\nabla \times (\nabla \times \boldsymbol{H}) = \nabla(\nabla \cdot \boldsymbol{H}) - \nabla^2 \boldsymbol{H} = -\nabla^2 \boldsymbol{H}. \tag{3}$$

Together with Maxwell's equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{4}$$

and the constitutive relation $\boldsymbol{B} = \mu \boldsymbol{H}$ equation (2) can now be written as

$$\nabla^2 \boldsymbol{H} = \sigma\mu \frac{\partial \boldsymbol{H}}{\partial t} + \varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2}. \tag{5}$$

This is the general wave equation. In a conducting medium, displacement currents can be neglected in comparison with conduction currents. This means that the last term in equation (5) can be omitted and the result is the magnetic diffusion equation

$$\nabla^2 \boldsymbol{H} = \sigma\mu \frac{\partial \boldsymbol{H}}{\partial t}. \tag{6}$$

For a long straight cylindrical conductor with radius a, and a time-dependent current, the problem can be solved via the magnetic field, equation (6) gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_\theta}{\partial r}\right) - \frac{H_\theta}{r^2} = \sigma\mu \frac{\partial H_\theta}{\partial t}, \tag{7}$$

in which a move to cylindrical coordinates and expansion of the equation (6) has been done. Since the total current in the cylinder is known, the relationship between the total current and the magnetic field on the surface of the conductor can be calculated via Biot-Savarts formula,

$$\mu H_\theta = B_\theta = \frac{\mu I}{2\pi r} \Rightarrow H_\theta(r = a) = \frac{I}{2\pi a}. \tag{8}$$

The relationship between magnetic field and current is given by equation (1) neglecting the displacement current, and in cylindrical coordinates the problem gives

$$j_z = \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) = \frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r}. \tag{9}$$

Especially equation (7) gives the basis of the first numerical experiments conducted so far, as the total current in the conductor is known, the magnetic field on the surface of the conductor is also known. Any change in the total current will immediately be reflected in a change in the magnetic field on the surface of the conductor.

There is an analytical solution to the problem, but only for harmonic currents, the proof can be found in chapter 30,4 in [1]. The solution for a harmonic current with angular frequency ω and amplitude I is

$$j_z = Re\left(\frac{Ik}{2\pi a}\frac{J_0(kr)}{J_1(ka)}e^{i\omega t}\right) \tag{10}$$

where $k = \sqrt{-i\mu\sigma\omega}$ and J_n are the Bessel functions.

This is under the assumption that σ, μ and ε are constant, this assumption is generally *not* true, especially when it comes to the conductivity, σ, that changes a lot when the temperature changes. When the material in the conductor heats up, its properties change. For the case where the properties changes, equation (2) has to be written

$$\nabla \times (\nabla \times \boldsymbol{H}) = (\nabla \times \sigma\boldsymbol{E}) + \frac{\partial}{\partial t}(\nabla \times \varepsilon\boldsymbol{E}). \tag{11}$$

The left-hand side becomes, as before,

$$-\nabla^2\boldsymbol{H}, \tag{12}$$

but the right-hand side becomes different, the first term becomes

$$\begin{aligned}(\nabla \times \sigma\boldsymbol{E}) &= \sigma(\nabla \times \boldsymbol{E}) + \nabla\sigma \times \boldsymbol{E}\\ &= -\sigma\frac{\partial(\mu\boldsymbol{H})}{\partial t} + (\nabla\sigma) \times \boldsymbol{E}\\ &= -\sigma\mu\frac{\partial \boldsymbol{H}}{\partial t} - \sigma\frac{\partial\mu}{\partial t} + (\nabla\sigma) \times \boldsymbol{E}\end{aligned}\quad , \tag{13}$$

and the second term becomes

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla \times \varepsilon\boldsymbol{E}) &= \frac{\partial}{\partial t}(\varepsilon(\nabla \times \boldsymbol{E})) + \frac{\partial}{\partial t}((\nabla\varepsilon) \times \boldsymbol{E})\\ &= -\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial(\mu(H))}{\partial t}\right) + \frac{\partial}{\partial t}((\nabla\varepsilon) \times \boldsymbol{E})\\ &= -\varepsilon\mu\frac{\partial^2\boldsymbol{H}}{\partial t^2} - \frac{\partial\varepsilon}{\partial t}\frac{\partial(\mu\boldsymbol{H})}{\partial t} + \frac{\partial}{\partial t}((\nabla\varepsilon) \times \boldsymbol{E}).\end{aligned} \tag{14}$$

Which gives the more general wave equation

$$\nabla^2\boldsymbol{H} = \sigma(t)\mu\frac{\partial\boldsymbol{H}}{\partial t} + \sigma\frac{\partial\mu}{\partial t} - (\nabla\sigma)\times\boldsymbol{E} + \varepsilon\mu\frac{\partial^2\boldsymbol{H}}{\partial t^2} + \frac{\partial\varepsilon}{\partial t}\frac{\partial(\mu\boldsymbol{H})}{\partial t} - \frac{\partial}{\partial t}((\nabla\varepsilon)\times\boldsymbol{E}). \tag{15}$$

This is a quite much more complicated equation than the one for constant material properties. But for a non-magnetic conductor the assumption can be made that the permittivity and the permeability are constant in both time and space and that they are equal to the values of vacuum. With these assumptions the last equation, equation (15), can be written as

$$\nabla^2 \boldsymbol{H} = \sigma(t)\mu \frac{\partial \boldsymbol{H}}{\partial t} + \varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - (\nabla\sigma) \times \boldsymbol{E}. \tag{16}$$

This equation has one extra term in comparison with equation (5), that term can be expanded to

$$(\nabla\sigma) \times \boldsymbol{E} = (\nabla\sigma) \times \left(\frac{1}{\sigma}\boldsymbol{j}\right) = (\nabla\sigma) \times \left(\frac{1}{\sigma}\left(\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t}\right)\right). \tag{17}$$

If, in equation (16), only a *time dependent* conductivity is considered the result is the equation

$$\nabla^2 \boldsymbol{H} = \sigma(t)\mu \frac{\partial \boldsymbol{H}}{\partial t} + \varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2}. \tag{18}$$

If again the effects of displacement current contribution is neglected in the equations (16) and (18) the result is

$$\nabla^2 \boldsymbol{H} = \sigma(t)\mu \frac{\partial \boldsymbol{H}}{\partial t} - (\nabla\sigma) \times \left(\frac{1}{\sigma}(\nabla \times \boldsymbol{H})\right) \tag{19}$$

and

$$\nabla^2 \boldsymbol{H} = \sigma(t)\mu \frac{\partial \boldsymbol{H}}{\partial t}. \tag{20}$$

Equation (20) is quite straight forward, it just adds the time dependence and nothing more. Equation (19) on the other hand needs more attention. The second term can, in cylyndrical coordinates and via cylinder symmetry, be expanded to the following

$$\frac{1}{\sigma}(\nabla \times \boldsymbol{H}) = \frac{1}{\sigma}\left(\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r}\right)\hat{\boldsymbol{\theta}} \tag{21}$$

and then, in cylyndrical coordinates and via cylinder symmetry,

$$\nabla\sigma = \frac{\partial \sigma}{\partial r}\hat{\boldsymbol{r}}. \tag{22}$$

Combined with equation (19), equation (21) and (22) gives,

$$\frac{\partial^2 H_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\theta}{\partial r} - \frac{H_\theta}{r^2} - \frac{1}{\sigma}\frac{\partial \sigma}{\partial r}\left(\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r}\right) = \sigma\mu\frac{\partial H_\theta}{\partial t}. \tag{23}$$

The three equations (7), (8) and (9) are linear differential equations and as such they are numerically solvable by standard methods. The equation (23) on the other hand is non-linear, since it depends on the temperature, which in turn depends on $\boldsymbol{H}$. This means that a numerical solution requires a more advanced algorithm.

2.2 Thermal distribution

As current flows in a conductor the conductor will heat up and the heat will spread out inside the conductor. One problem to solve is how fast the heat spreads inside the conductor. Can the temperature in the conductor be viewed as uniformly distributed? The spread of heat inside the conductor is described by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{\kappa h}{\lambda}. \tag{24}$$

Where h is a source of heat and $\kappa = \frac{\lambda}{c_p \rho}$, where λ is thermal conductivity, c_p specific heat capacity and ρ density. In cylindrical coordinates and with the use of symmetry equation (24) becomes.

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right) + \frac{\kappa h}{\lambda}. \tag{25}$$

This expands to

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{h}{\lambda} \right). \tag{26}$$

The boundary conditions is of Neuman type ie. $\frac{\partial T}{\partial r} = 0$ at $r = a$ and $r = 0$. For the edge of the conductor this is motivated by neglecting heat transfer to the surroundings by radiation or conduction and in the center by symmetry.

To get some sort of idea of how fast the thermal conduction inside the conductor is, this equation has to be solved. The solution will be calculated primarily for copper. Equation (26) is a linear partial differential equation and can be solved with standard numerical methods or quite often even analytically.

2.3 Combining the current and thermal distributions

Finding the relation between the heating effect of the current, the thermal effect on the electric conductivity and the temperature, it is necessary to calculate the thermal and current distributions at work inside the same system.

The relation between electric resistivity and conductivity is

$$\rho = \frac{1}{\sigma} \tag{27}$$

and

$$\sigma = \frac{1}{\rho_0 (1 + \alpha (T - T_0))}. \tag{28}$$

Here ρ_0 is the electric resistivity at room temperature and α is the temperature coefficient of the material of the conductor. Now equation (20) can be written as

$$\nabla^2 \boldsymbol{H} = \frac{1}{\rho_0(1+\alpha(T(t)-T_0))}\mu\frac{\partial \boldsymbol{H}}{\partial t}. \tag{29}$$

Equation (23) can also be rewritten, the result is as follows

$$\begin{aligned} &\frac{\partial^2 H_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial H_\theta}{\partial r} - \frac{H_\theta}{r^2} - \\ &\rho_0(1+\alpha(T(t)-T_0))\frac{\partial}{\partial r}\left(\frac{1}{\rho_0(1+\alpha(T(t)-T_0))}\right)\left(\frac{\partial H_\theta}{\partial r} + \frac{H_\theta}{r}\right) = \\ &\frac{1}{\rho_0(1+\alpha(T(t)-T_0))}\mu\frac{\partial H_\theta}{\partial t}. \end{aligned} \tag{30}$$

At the same time the heat equation for the temperature distribution T of the conductor is, with added heating from current flowing in the conductor, h is $\frac{\boldsymbol{j}^2}{\sigma}$. This gives

$$\frac{\partial T}{\partial t} = \kappa\nabla^2 T + \frac{\kappa \boldsymbol{j}^2}{\sigma(t)\lambda}. \tag{31}$$

Here is $\sigma(t)$ defined as in equations (27) and (28). Expanding this according to equation (26) gives the following result

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\boldsymbol{j}^2\rho_0(1+\alpha(T-T_0))}{\lambda}\right). \tag{32}$$

Equation (31) is a non-linear differential equation, which is hard to solve numerically, but we will propose a number of methods to get around that to get a close approximation.

2.4 Material and parameters

This section covers the material primarily studied, copper, and the parameters connected to it. The numerical values of the properties are taken from [2], chapter T-1.1 and T-2.1 .

$T_m = 1356$	[K]	melting temperature
$T_v = 2855$	[K]	vaporizing temperature
$\alpha = 4.33 * 10^{-3}$	[1 / K]	Temperature coefficient
$c_p = 385$	[J / kg K]	Specific heat capacity
$\lambda = 400$	[W / m K]	Thermal conductivity
$l = 205 * 10^3$	[J / kg]	Heat of fusion
$d = 8.96 * 10^3$	[kg / m^3]	density
$\sigma_0 = 5.961 * 10^7$	[1/ Ω m]	Conductivity at room temperature

3 Numerical implementation

This section will be an application of the formulas derived in the earlier sections. It contains the same division in three subsections. One section for the current distribution, one section for the thermal distribution and one section for the combination of these two properties. Throughout this section the following definitions will be used.

H	$[A/m]$	Magnetic field strength (magnetic intensity), θ-wise
T	$[K]$	Temperature
$\boldsymbol{D}, \boldsymbol{F}$		Difference matrices
j	$[A/m^2]$	Current density, z-wise
N		number of points in the radius, numbered from 1 to N
n		Time step index
i		Radius index
a	$[m]$	total radius of conductor
r	$[m]$	present radius
h	$[m]$	Δr
k	$[s]$	Δt

3.1 Current distribution

3.1.1 Numerical scheme 1

To find the current distribution a partial differential equation must be solved. Starting with equation (6), where σ constant in time and space will simplify the problem.

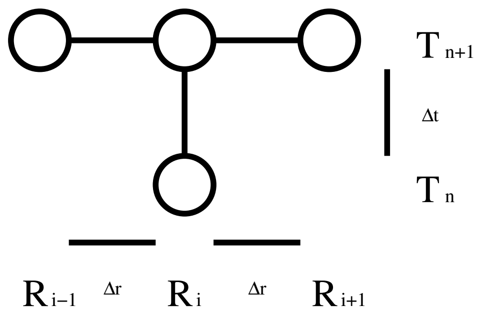

Figure 1: The difference scheme in Magnetic field calculation.

For the solving of the current distribution in time and space numerically a backward difference scheme is chosen. The scheme implemented in this section is a standard backward difference scheme, illustrated in figure 1, the advantage of this scheme over any forward difference scheme is that it is always stable. This difference scheme results in the following matrix

$$\boldsymbol{D} = \begin{pmatrix} c+\frac{h^2}{r^2} & -(1+\frac{h}{2r}) & & & \mathbf{0} \\ & \ddots & \ddots & & \\ & -(1-\frac{h}{2r}) & c+\frac{h^2}{r^2} & -(1+\frac{h}{2r}) & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & -(1-\frac{h}{2r}) & c+\frac{h^2}{r^2} \end{pmatrix} \tag{33}$$

where $c = 2 + \frac{h^2}{k}$. The matrix $\boldsymbol{D}$ is a tri-diagonal matrix with as many rows as the spatial resolution, N. The boundary conditions for the difference scheme given by equation (8) and the fact that in the centre of a cylindrical conductor the magnetic field is zero. Then there is the system $\boldsymbol{D}\boldsymbol{H}_{n+1} = \boldsymbol{H}_n$ to solve where $\boldsymbol{H}$ is a vector of the magnetic field strength on different distances from the centre of the conductor. The values at the edges, where the radius r is either equal to 0 or a, are already known. The value at r is 0, because there is no enclosed current so the magnetic field must be 0.

The current density in the z-direction is then calculated with equation (9) approximated with finite differences,

$$j_i = \frac{H_{i+1} - H_{i-1}}{2h} + \frac{H_i}{r} \tag{34}$$

For $r = 0$ the limit $r \to 0$ is used,

$$j_1 = 2\frac{-3H_1 + 4H_2 - H_3}{2h}$$

3.1.2 Numerical result 1

Starting with the total magnetic field equal to zero and then changing the total field at the outer edge yields an estimation of the field distribution inside the conductor.

To get an estimation of how large the error in this estimation is, it has to be compared to an analytically deduced solution for an harmonic current. This far in the numerical calculations there is an estimation error in the range of 1%. This error seems to be quite small, but the true effects on thermal distribution and electric conductivity are not known.

One way to decrease this error is to increase the resolution in time. The above implemented scheme has a truncation error of the order $\mathcal{O}(k + h^2)$. Where k is the time step and h is the spatial step. So increasing the resolution in time would only decrease the error linearly. By using another difference scheme the truncation error can be lowered to the order of $\mathcal{O}(k^2 + h^2)$. The difference scheme now looks like figure 2 and is called a Crank-Nicolson difference scheme.

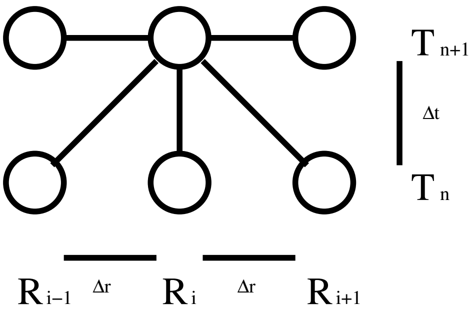

Figure 2: The improved difference scheme in the Magnetic field calculation.

3.1.3 Numerical Scheme 2

The numerical method used for solving equation (7) and (23) is the finite differences method with the Crank-Nicolson scheme, illustrated in figure 2. The idea behind Crank-Nicolson is that for each point that is to be calculated, the neighbouring points in both the new time step and old step are used to approximate spatial derivatives. Since the solution of points in the new time step are unknown the method is implicit which requires that a system of equations is solved in each time step. The advantages of this scheme are that it is unconditionally stable and has truncation error of $\mathcal{O}(k^2 + h^2)$. The equation for H_θ in spatial point i and time point $n+1$, using equation (7) becomes with the θ subscript dropped, derivatives approximated with finite differences, h as spatial step (N spatial points) and k as time step

$$\begin{aligned}
&\frac{H_{i+1}^{n+1} - 2H_i^{n+1} + H_{i-1}^{n+1}}{2h^2} + \frac{H_{i+1}^{n} - 2H_i^{n} + H_{i-1}^{n}}{2h^2} \\
&+ \frac{H_{i+1}^{n+1} - H_{i-1}^{n+1}}{4hr} + \frac{H_{i+1}^{n} - H_{i-1}^{n}}{4hr} - \frac{H_i^{n+1}}{2r^2} - \frac{H_i^{n}}{2r^2} \\
&= \sigma\mu \frac{H_i^{n+1} - H_i^{n}}{k} \qquad \Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{h}{4r}\right) H_{i-1}^{n+1} - \left(1 + \frac{h^2}{2r^2} + \frac{\sigma\mu h^2}{k}\right) H_i^{n+1} + \left(\frac{1}{2} + \frac{h}{4r}\right) H_{i+1}^{n+1} \\
= -&\left(\frac{1}{2} - \frac{h}{4r}\right) H_{i-1}^{n} + \left(1 + \frac{h^2}{2r^2} - \frac{\sigma\mu h^2}{k}\right) H_i^{n} - \left(\frac{1}{2} + \frac{h}{4r}\right) H_{i+1}^{n}.
\end{aligned}$$

At the boundary is the solution known, so the boundary values are not calculated but they will affect the solution in two different ways. At the centre point, $r = 0$, the value is zero and will not have an effect on the solution of the system of equations. However, at the edge of the conductor, $r = a$, the H-field is not zero. This is dealt with by modifying the right-hand side in the last equation. The system of equations can in matrix form be written as

$$\boldsymbol{D}H^{n+1} = -\boldsymbol{D}H^{n} - \frac{2\sigma\mu h^2 H^n}{k}$$

with

$$\left(1 + \frac{h}{4r_{N-1}}\right)\left(H_N^{n+1} + H_N^n\right)$$

subtracted from the right-hand side of the last equation due to the boundary condition at the edge. And the matrix

$$\boldsymbol{D} = \begin{pmatrix} 1+c_2^n & -\frac{1}{2}+\frac{h}{4r} & & & \mathbf{0} \\ & \ddots & \ddots & & \\ & -\frac{1}{2}+\frac{h}{4r} & 1+c_i^n & -\frac{1}{2}-\frac{h}{4r} & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & -\frac{1}{2}+\frac{h}{4r} & 1+c_{N-1}^n \end{pmatrix} \tag{35}$$

with $c_i^n = \frac{h^2}{2r^2} - \frac{\sigma\mu h^2}{k}$, as result.

For equation (23) D will be different for each time step, because the extra term in the equation has parameters that are time dependent. The system of equations will now be written like

$$\boldsymbol{D}^{n+1}H^{n+1} = -\boldsymbol{D}^{n}H^{n} - \frac{2\sigma^n\mu h^2 H^n}{k}$$

where the superscripts of $\boldsymbol{D}$ and σ state the time step and

$$\left(\frac{1}{2} + \frac{h}{4r_{N-1}}\right)\left(\frac{1}{\sigma_{N-1}^{n+1}}\frac{\partial\sigma_{N-1}^{n+1}}{\partial r}\frac{h}{4}H_N^{n+1} + \frac{1}{\sigma_{N-1}^{n}}\frac{\partial\sigma_{N-1}^{n}}{\partial r}\frac{h}{4}H_N^{n}\right)$$

subtracted from the right-hand side of the last equation due to the boundary condition at the edge. And the matrix

$$\boldsymbol{D^n} = \begin{pmatrix} -1+c_2^n & \frac{1}{2}+\frac{h}{4r}+d_2^n & & & \mathbf{0} \\ & \ddots & \ddots & & \\ & \frac{1}{2}-\frac{h}{4r}-d_i^n & -1+c_i^n & \frac{1}{2}+\frac{h}{4r}+c_i^n & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \frac{1}{2}-\frac{h}{4r}-d_{N-1}^n & -1+c_{N-1}^n \end{pmatrix} \tag{36}$$

with $d_i^n = \frac{1}{\sigma_i^{n+1}}\frac{\partial\sigma_i^{n+1}}{\partial r}\frac{h}{4}$ and $c_i^n = \frac{h^2}{(r_i)^2} - \frac{\sigma_i^n \mu h^2}{k} - \frac{\sigma_i^n h^2}{\sigma r_i} +$.

3.1.4 Numerical result 2

For error analysis the analytical solution for a time harmonic current has been used. This can be seen in figure 3. In figure 4 the relative difference between the numerical and the analytical solutions along the radius is plotted. Where the numbers are the number of points in time and space. The error tends to decrease by a factor of four when the step length is halved which is in accordance with the theory of the error for the Crank-Nicolson scheme. This makes it safe to assume that the implementation of the numerical method is correct.

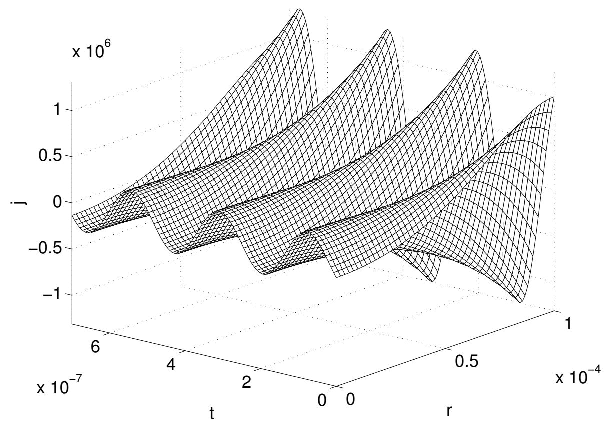

Figure 3: The current density distribution from a harmonic current.

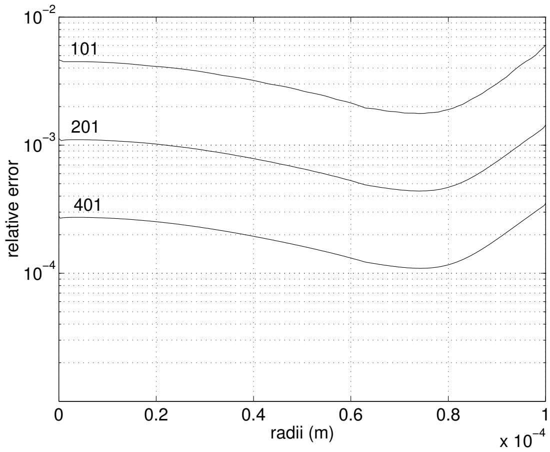

Figure 4: Error of numerical solution for a harmonic current.

3.2 Thermal distribution

3.2.1 Numerical scheme

To find the temperature distribution, equation (32) is solved with the same method that is used for the current distribution. The boundary condition here is of neuman type which is modeled by, in the first and last equation, setting $T_{-1} = T_1$ and $T_{N-1} = T_{N+1}$ respectively. As heat source the average current density of both time steps (n and $n+1$) is used. The system of equations can then be written as

$$\boldsymbol{F}T^{n+1} = -\boldsymbol{F}T^{n} - \frac{2h^2}{k\kappa}T^n - \frac{h^2(j^{n+1}+j^n)^2}{4\sigma\lambda},$$

according to [2] and section 2.4 , the constants for copper are

$$\begin{aligned} c_p &= 385 \text{ specific heat capacity} \\ \lambda &= 400 \text{ thermal conductivity} \\ d &= 8.96 * 10^3 \text{ density} \\ \kappa &= \frac{\lambda}{d * c_p} \\ c &= -1 - \frac{h^2}{k * \kappa} \end{aligned}$$

The difference scheme applied for solving the temperature distribution over time and space results in the following difference matrix

$$
\boldsymbol{F} = \begin{pmatrix} c & 1 & & & \mathbf{0} \\ & \ddots & \ddots & & \\ & (\frac{1}{2} - \frac{h}{4r}) & c & (\frac{1}{2} + \frac{h}{4r}) & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & 1 & c \end{pmatrix} \tag{37}
$$

The matrix $\boldsymbol{F}$ is a tri-diagonal matrix with as many rows as the number N and to solve equation (26) with the above algorithm some initial temperature distribution is needed. Figure 5 is an example of a solution.

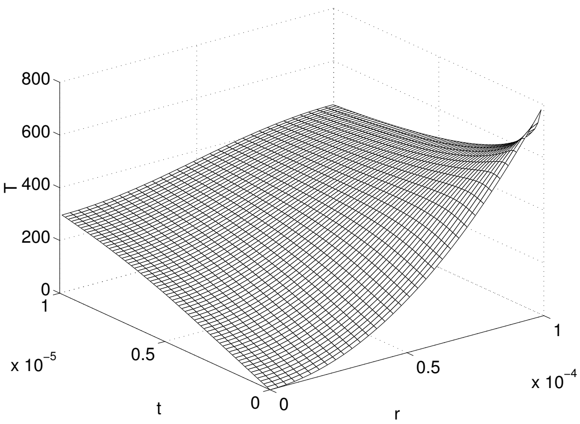

Figure 5: The temperature diffusion.

3.2.2 Result

The initial results suggests that the thermal distribution may be slow enough to be a factor in the current distribution. The conductor might be heated in a non-uniform distribution and thus have a non-uniform distributed electrical conductivity. Figure 5 shows the temperature spreading out inside the conductor, this figure was created with no external heating, i.e. $\boldsymbol{j} = 0$ and calculated in an environment where the radius is $100 * 10^{-6}$ m and the total time is $1 * 10^{-5}$ seconds. The initial distribution of heat along the radius r is based on the formula $T_r = 800\frac{r^2}{a^2}$ where a is the total radius.

3.3 Combining the current and thermal distributions

This section will cover the complete implementation of the numerical implementation of the current distribution problem. To solve the non-linear equations derived, two different methods were suggested.

One method is to increase the resolution in time for the thermal distribution. And then calculate the thermal distribution twice as many times as the current distribution or more. The current distribution would then be calculated on the basis of the last known current distribution, in time n, and the thermal distribution between time $n+1$ and n. The thermal distribution in time $n+1$ would then be calculated based on the thermal distribution between time n and time $n+1$ and the current distribution in time $n+1$.

Another method is to approximate the current distribution by a p-th degree polynomial and use that as a approximation for calculating the thermal distribution in time step $n+1$. The current distribution would then be calculated on the basis of the approximated thermal distribution in time step $n+1$ and the current distribution in time step n

The method chosen here is the p-th degree polynomial approximation with p equal to 2. In time step n, approximate the current distribution in the following time step $n+1$ by using a second-degree polynomial approximation. That allows a calculation of an expected value for the current density. That approximation is used for calculating the thermal distribution. Then calculation of the actual current density based on the previous time step n and the thermal distribution in time step $n+1$ is done.

3.3.1 Numerical schemes and why

In this part of connecting the current and thermal distributions, the heating effect, the changing material properties and the thermal transport are all implemented.

The calculations are done in the following order:

1. $b = -\boldsymbol{D}^n H^n - \frac{2h^2\sigma\mu}{kH^n}$
2. $jp = j^{n-2} - 3j^{n-1} + 3j^n$, the polynomial approximation
3. $f = -\boldsymbol{F}T^n - \frac{2h^2}{k\kappa}T^n - \frac{h^2(jp+j^n)^2}{4\sigma\lambda}$
4. $T^{n+1} = \boldsymbol{F}^{-1}f$
5. $\sigma = \frac{\sigma_0}{1+\alpha(T^{n+1}-T^0)}$, recalculation of σ
6. $\frac{\partial\sigma}{\partial r}_i = \frac{\sigma_{i+1}-\sigma_{i-1}}{2h}$
7. uppdate $\boldsymbol{D}^{n+1}$

8. $H^{n+1} = (\boldsymbol{D}^{n+1})^{-1}b$

9. $j_i^{n+1} = \frac{H_{i+1}^{n+1} - H_{r-1}^{n+1}}{2h} + \frac{H_i^{n+1}}{r_i}$

The updating of $\boldsymbol{D}^{n+1}$ is done according to (36) because $\boldsymbol{D}$ contains time dependency. The calculation of the temperature and current distributions stop when the temperature reaches the vaporisation temperature. The polynomial approximation is done using Lagrange's interpolation formula. Using three equally spaced points this yields the approximation in the point 2 in the list.

3.3.2 Result from the process

For initial testing of this implementation a copper wire with radius $60 * 10^{-6}$ m and a current pulse with top value $3*10^3$ A that has a duration of $0.2*10^{-6}$ s was used. This input generates a close to uniform distribution of both temperature and current. For a conductor with large radius the distributions will look different. This subsection will show how the distribution of current and temperature changes when different input currents are given. The pulse given as input has the form

$$i(t) = -I_0 \frac{cos(2 * \pi * \frac{t}{\Delta t}) - 1}{2} \tag{38}$$

where Δt is the desired length of the pulse, and t goes from 0 to Δt. I_0 is the desired top value. The equation (38) will be used to generate input in all the following experiments.

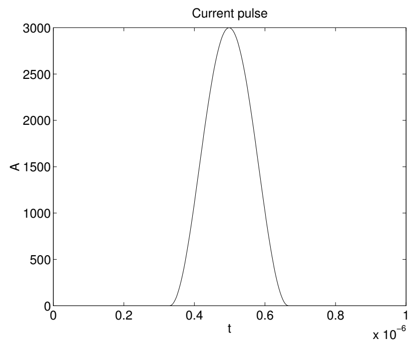

Figure 6: The current pulse as input.

The figures 6 and 7 describes in turn the current transient that is the input to the system, the resulting current density distribution and the resulting temperature distribution. A note about the figures 6 and 7 is that they describe an implementation of the equations (29) and (32). This implementation lacks some terms that will have a significant impact on the

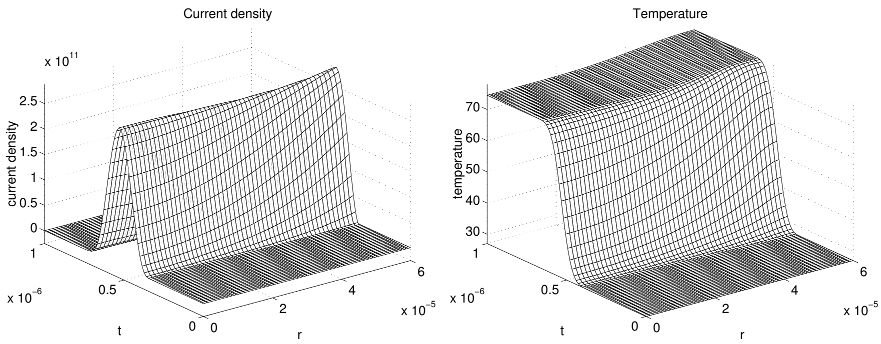

Figure 7: The resulting current density and temperature distributions.

final distributions in many cases. However this input is not one of the more extreme cases.

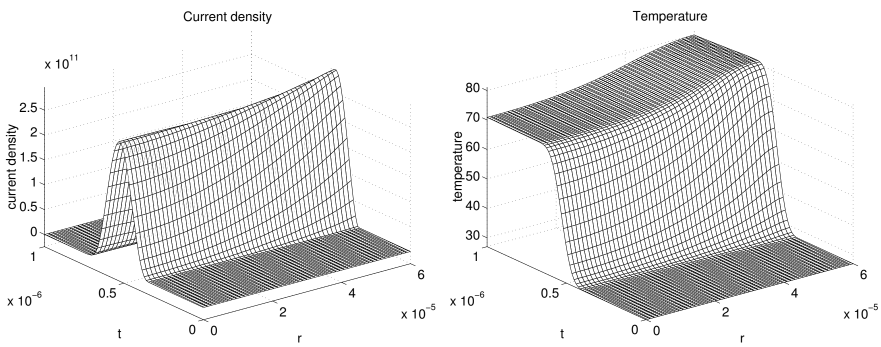

Figure 8: The resulting current density and temperature distributions, from equation 30.

The figure 8 is calculated with the same input as figure 7, the current pulse in figure 6, but an implementation of the equation (30). As it can be seen, the difference is not very large. The reason to why the figures look similar, is that the current density is close to uniformly distributed. The close to uniform distribution means that the extra term in equation (30) compared to equation (29) gives a contribution that is close to zero. This in turn means that the results will not differ much. Especially the current density distribution is almost identical while the temperature distribution is somewhat more different.

3.3.3 Modeling melting and vaporising

When the temperature in the conductor reaches the melting point it does not instantly melt. The melting process require some additional energy. This will make the temperature remain at the melting temperature for some time while the necessary energy to melt the conductor is absorbed. The same also goes for the transition to gas form.

In the algorithm this is modeled by keeping track of how much material that are melted in each point and if the temperature rises above the melting point and the point is not completly melted, the excess temperature is converted to heat and used to melt more of the point instead of heating the point over the melting point. The same goes for the opposite process. If vaporising temperature is reached the algorithm stops since it can not be considered to have much accuracy any more and the conducting wire probably is about to explode.

4 Final results

One of the results of the tests, is that for some of the inputs the current is close to uniformly distributed. For the time-intervals and the current pulses primarily studied, it can also be seen that the temperature *does* give some difference in how the current is distributed, however not much. One of the reasons for this result, is that the temperature is very dependent on the heating from the current flowing through the conductor. When the current is close to uniformly distributed so is the heating effect and the temperature becomes almost uniformly distributed as well.

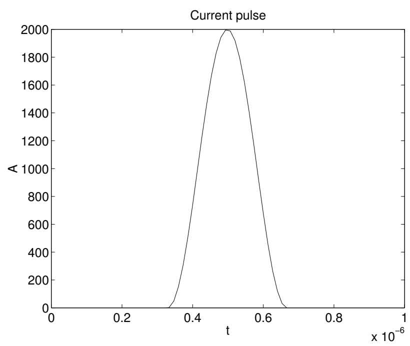

Figure 9: The current pulse.

This is valid for current transients with top value of $2*10^3$ A and duration of about $0.2*10^{-6}$ s and radius of $50*10^{-6}$ m in figure 9 that is given by the equation (38). The result of these parameters is shown in figure 10.

Increasing the radius of the conductor, the current distribution becomes different, more current flows close to the surface. The heating effect of the current makes the conductor heat up on the surface faster than closer to the centre of the conductor. In figure 11 the current pulse is the same as the current pulse in figure 6, but the resulting current and heat distributions are different.

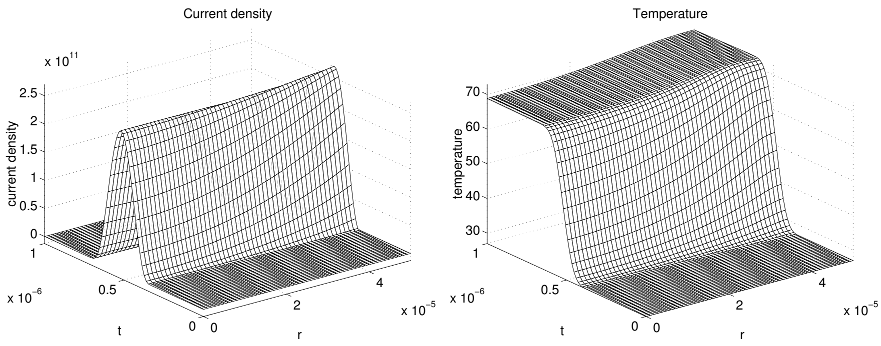

Figure 10: Current density and temperature distributions.

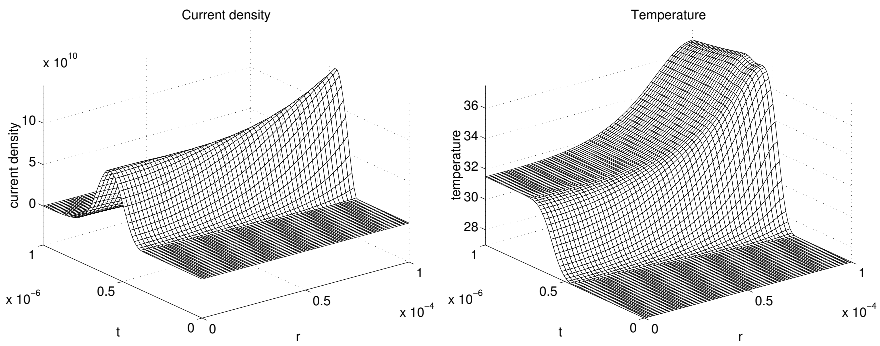

Figure 11: Current density and temperature distributions.

Figure 11 describes what happens to the temperature and current distributions for a conductor with radius $100 * 10^{-6}$ m but otherwise the same conditions as before.

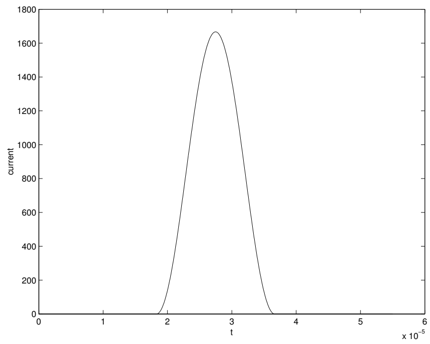

Figure 12: The original current pulse.

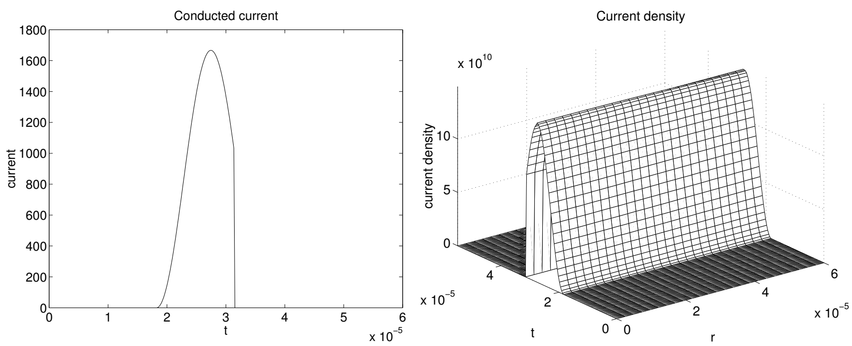

Figure 13: The actual conducted current and resulting current density.

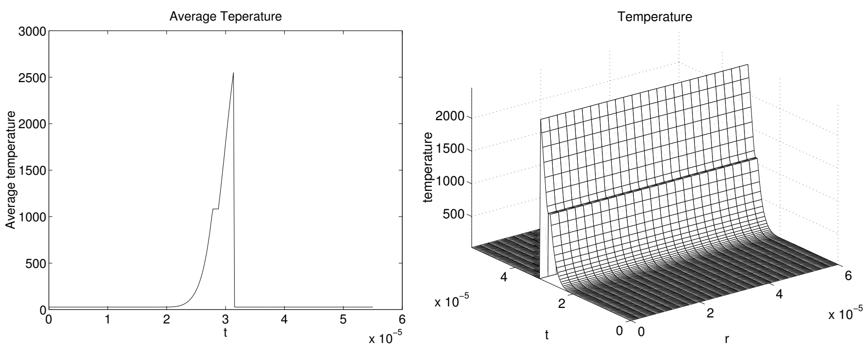

Figure 14: The average temperature and the temperature distribution.

What also was tested was a conductor with radius $60 * 10^{-6}$ m and a top current at $1.6 * 10^3$ A and a pulse length of about $15 * 10^{-6}$ s, this showed that the current and temperature was distributed in a close to perfect uniform distribution. This can be seen in figure 13 and figure 14. The final average temperature is close to 2600 °C which is almost exactly the boiling temperature at 2855 °K for copper. The stop at the melting temperature can also be seen in the figure. As the calculations stop at the limit temperature, which is set to the boiling temperature, we can then draw the conclusion that any non-uniformities in the current distribution can be neglected in this case.

5 Conclusions

What has been found is that the *distribution* of the current and temperature might not be relevant in conductors with a radius of $60 * 10^{-6}$ m and above. This is when a gaussian current pulse of top value less than $2 * 10^3$ A and duration shorter than $15 * 10-6$ s is applied. The current and temperature becomes almost perfectly uniformly distributed. To get any other distribution the duration of the pulse must be about 10 times shorter or the radius must be smaller. The only effect of increasing the top value of the pulse is that the temperature rises faster and the effect of increasing the radius is that the temperature rises slower. Another conclusion is that the distribution of the temperature *is* relevant, at least in those cases where there is a non-uniform current distribution. The speed with which the thermal energy spreads in copper is not large enough to estimate to infinity and it is not slow enough to estimate to zero.

A Origin of the report

This report was written as an assignment in the course "Tekniskt-vetenskapliga datorberäkningar" given on the department of scientific computing, Uppsala University, Sweden, spring term of 2003. The assignment was initiated by Anders Larsson at FOI, the Swedish Defence Research Agency.

B Thanks to

Anders Larsson, associate professor, FOI – Swedish Defence Research Agency

References

[1] E. Hallén. *Elektricitetslära.* Almquist & Wiksells, Uppsala, 1953.

[2] Carl Nordling and Jonny Österman. *Physics Handbook for science and Engineering.* Studentlitteratur, Lund, fifth edition, 1996.