UPPSALA UNIVERSITET

PROJECT REPORT

Sensitivity of optimal solutions for hot rolling with MATLAB

Deniz Kennedy

Project in Computational Science: Report

February 2014

Department of Information Technology

Abstract

Hot rolling is a metal forming process that is used in shaping certain large pieces of metal such as slabs, blooms, and billets to produce sheet metal. These metal slabs are usually heated up to $325-450^{\circ}$C depending on the type of the metal. They passed through specific rolling mills accordingly to the desired products and finally, they are hardened while the temperature decrease below the recrystallization temperature. The entire manufacturing process and the optimal production settings are desired. In order to investigate how to reduce the cost, a mathematical model for the process has been implemented in MATLAB at the company ABB. The model solves a large-scale nonlinear optimization problem and it is used as a black box. Sensitivity analysis is needed in the model to change certain physical model parameters that optimize the production process, which is the main purpose of this paper. Modifications of these specific physical parameters of the model in process allow to understand their impact on the cost. The sensitivity study is performed using *fmincon* and *MultiStart* solvers in MATLAB and certain results are obtained for different algorithms in *fmincon* and objective functions.

Keywords: optimization, MATLAB, sensitivity analysis, Lagrange multipliers

Contents

List of notations used in the paper

- *obj63*: Objective function describing specific power
- *obj71*: Objective function describing grain size
- *obj63AS0*: Objective function for specific power solved with the local algorithm *active-set* and nonlinear equality constraints are NOT substituted by the pair of nonlinear inequality constraints
- *obj63AS1*: Objective function for specific power solved with the local algorithm *active-set* and nonlinear equality constraints are substituted by the pair of nonlinear inequality constraints
- *obj63SQP0*: Objective function for specific power solved with the local algorithm *sqp* and nonlinear equality constraints are NOT substituted by the pair of nonlinear inequality constraints
- *obj63SQP1*: Objective function for specific power solved with the local algorithm *sqp* and nonlinear equality constraints are substituted by the pair of nonlinear inequality constraints
- *obj71AS0*: Objective function for grain size solved with the local algorithm *active-set* and nonlinear equality constraints are NOT substituted by the pair of nonlinear inequality constraints
- *obj71AS1*: Objective function for grain size solved with the local algorithm *active-set* and nonlinear equality constraints are substituted by the pair of nonlinear inequality constraints
- *obj71SQP0*: Objective function for grain size solved with the local algorithm *sqp* and nonlinear equality constraints are NOT substituted by the pair of nonlinear inequality constraints
- *obj71SQP1*: Objective function for grain size solved with the local algorithm *sqp* and nonlinear equality constraints are substituted by the pair of nonlinear inequality constraints

1 Introduction

1.1 Hot Rolling

Hot rolling is a process that creates individual parts, assemblies, or large-scale structures by moving metal slabs heated up to a very high temperature through several specific rolling mills. As Figure 1 shows, the process works in the following order. First, metal slabs are heated up in furnaces. Second, the metal slabs pass through several mills, which are roughing mills and finishing mills, respectively. These mills give a particular shape to the metal slabs by arranging the speed and the gap of the rolls. Third, the metal sheets are cooled at run-out table.

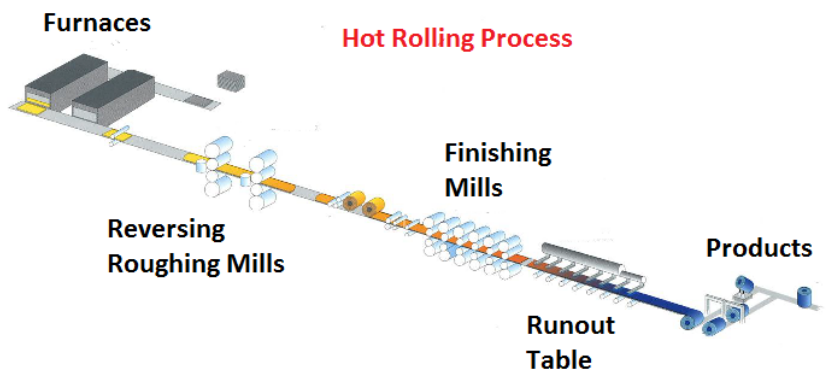


Figure 1: Hot Rolling Process

Since the process is expensive, a mathematical optimization model is implemented to simulate the hot rolling process prior to real production. The simulation model is implemented by the company ABB and provided as a black box. The aim of the research is to investigate the quality of the so-obtained optimal solutions using sensitivity analysis.

1.2 Optimization Model

The model is a constrained nonlinear optimization problem, described as

$$\begin{aligned} \underset{X}{\text{minimize}} \quad & F(X) \\ \text{subject to} \quad & g_i(X) \leq 0, \; i = 1, \ldots, 130 \\ & h_j(X) = 0, \; j = 1, \ldots, 10 \\ & lb \leq X \leq ub \end{aligned} \tag{1.1}$$

where X is the vector of 33 optimization variables such as material speed, inter-pass tensions, roll gaps, roll speeds, and furnace temperature. *F(X)* is a nonlinear, non-smooth and non-convex objective function, g_i*(X)* is a set of nonlinear inequality constraints, $h_j(X)$ is a set of nonlinear equality constraints, *lb* and *ub* are lower and upper bounds for the variables, respectively.

1.3 Theory

In order to find a local solution for the problem, the so-called first-order necessary conditions have to be satisfied. First, the Lagrangian function is defined as

$$\mathcal{L}(X;\lambda,\mu) = F(X) + \sum_{i=1}^{130} \lambda_i g_i(X) + \sum_{j=1}^{10} \mu_j h_j(X), \tag{1.2}$$

where λ is the vector of Lagrange multipliers for the nonlinear inequality constraints and μ is the vector of Lagrange multipliers for the nonlinear equality constraints.
Let X^* be a local solution for the problem with the vectors of Lagrange multipliers λ^* and μ^* at the minimum point. The necessary conditions are

$$\begin{aligned}
&\nabla_X \mathcal{L}(X^*;\lambda^*,\mu^*) = 0, && && (1.3a)\\
&g_i(X^*) \leq 0, && for\ i = 1,\dots,130 && (1.3b)\\
&h_j(X^*) = 0, && for\ j = 1,\dots,10 && (1.3c)\\
&\lambda_i^* \geq 0, && for\ i = 1,\dots,130 && (1.3d)\\
&\lambda_i^* g_i(X^*) = 0, && for\ i = 1,\dots,130 && (1.3e)
\end{aligned}$$

where the conditions (1.3e) are complementarity conditions that imply which nonlinear inequality constraint is active. The minimum point is acquired by satisfying the conditions (1.3), which are also known as KKT conditions[1, 2].

2 Methods

2.1 MATLAB

In MATLAB, there are several different approaches to solve the optimization problem; however the *fmincon* solver is used as a main tool. Moreover, the *MultiStart* solver is used to solve the same problem with many different initial points.

2.1.1 *fmincon*

The *fmincon* solver is one of the tools that solves local nonlinear constrained optimization problems. The solver requires an objective function, bound limits on variables, constraints and an initial point accordingly to model (1.1) [1]. This solver has different algorithms and each of them is a way to solve the problem[2]. The algorithms *sqp* and *active-set* are used in this paper. The results are returned by the solver at the end of the calculations.

[1] More information about *fmincon*: http://www.mathworks.se/help/optim/ug/fmincon.html
[2] More information about the algorithms: http://www.mathworks.se/help/optim/ug/constrained-nonlinear-optimization-algorithms.html

2.1.2 *MultiStart*

The *MultiStart* solver is the method to solve an optimization problem starting with many different initial points. Since the optimization problem is non-convex, it could have several local solutions. Different local solutions could be found by the optimization routine, starting from different initial points and obtaining different local solutions for different initial points. The optimal objective function value and solution vector are given as ranked lists of the best found objective function values and correspondent optimal points. The chosen problem for the solver is model (1.1) and parallelization is also available to reduce the execution time[3].

2.2 Feasible Set

A set of initial points is generated by using the *MultiStart* solver in order to separate the infeasible initial points leading to a feasible solution from the ones not leading to a feasible solution. Since the model is very strict in constraints, the next step is to find the initial points that give a feasible solution after running the *fmincon* solver. It works as follows. First, the random initial points are generated by *MultiStart* by satisfying only the lower and upper bound constraints. These initial points do not satisfy many of the nonlinear inequality constraints and especially the nonlinear equality constraints. Second, these initial points are triggered in *fmincon* solver and finally, if they lead to a feasible solution, they are saved and used in the future analysis. An example is given in Figure 2 (*Warning*: The real problem is thirty three dimensional which is impossible to picture. All the figures in the section ***Methods*** are plotted in two dimensions.)

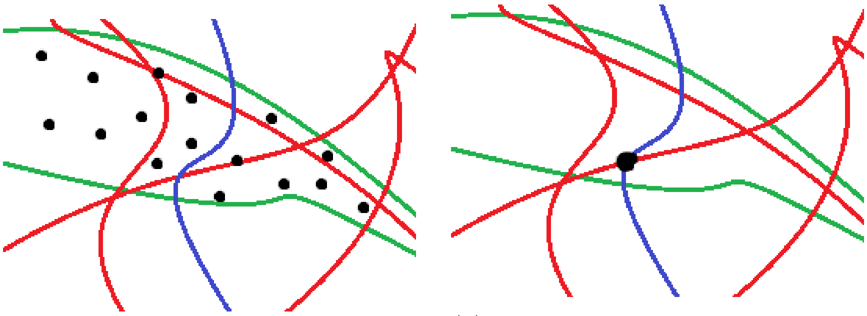
2.3 Lagrange Multipliers and Active Set

The solution of the optimization problem is terminated when it reaches a minimum value subject to the given constraints. If the solution belongs to the boundary of some constraint, such constraints are called active and the impact of these active constraints on the solution could be found by acquiring the Lagrange multiplier associated with these constraints. The larger Lagrange multiplier is, the more impact it has on the solution. Since the solution is optimal, the Lagrange multipliers associated with inequality constraints have nonnegative values. The combination of all active constraints is called an active set. Note that the Lagrange multipliers can be acquired for inequality constraints, because the equality constraints are always active by the definition and their constraint value cannot change[1, 2]. An example is given in Figure 3.

2.4 Constraints Modifications

To understand how sensitive is the optimal solution with respect to the small change in the right-hand side of the active constraints, the constraints are perturbed and the new so-obtained optimization problem is solved. The inequality constraints and equality constraints are modified in different ways.

[3] More information about *MultiStart*:http://www.mathworks.se/help/gads/multistartclass.html



(a) The initial points generated by *MultiStart* before running *fmincon*

(b) The feasible solutions, which have the same solution, generated after running *fmincon*

Figure 2: An example of extracting the initial points that lead to a feasible solution from the infeasible initial points. The green curves are upper and lower bound constraints, the red curves are nonlinear inequality constraints (referring to one of the two areas separated by the curve in the plane), the blue curve is nonlinear equality constraint, and the black points are initial points. Note that all 16 black points in (a) are infeasible due to their positions, i.e. they are not on the equality constraint and some of them are not feasible due to the inequality constraints. However 14 of them lead to a feasible solution , in which there is a single optimum, by using the *fmincon* solver in (b), which are reserved for future analysis, and two of them fail to lead to a feasible solution, which are eliminated.

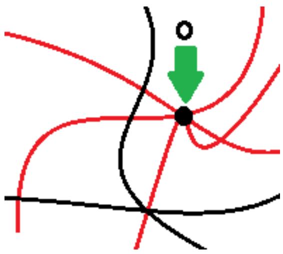


Figure 3: An example of the active set. The red curves are active whereas the black curves are inactive. All of these curves can be any kind of constraints except equality constraints as long as the points are are feasible with respect to the lower and upper bound constraints. One of the initial point (the empty point) that leads to a feasible solution generated by the *MultiStart* is obtained by *fmincon* at the constraints (the filled point). In this case, these red curves are the limiters of the problem and they give a positive Lagrange multiplier, whereas the inactive curves give zero Lagrange multiplier.

2.4.1 Perturbation of the Inequality Constraints

When the active set is determined, the active constraints are separated into groups: active lower bound constraints, active upper bound constraints, and active nonlinear inequality constraints. The largest Lagrange multiplier in the active set is chosen to perturb its constraint, because the largest Lagrange multiplier has the largest impact on the solution.
If the largest Lagrange multiplier is associated with one of the active lower bound constraints, the constraint is changed to a smaller value by subtracting 0.01 from the constraint value as in (2.1) in order to extend the feasible region.

$$lb \leq X \Rightarrow lb - 0.01 \leq X \tag{2.1}$$

If the largest Lagrange multiplier is associated with one of the active upper bound constraints, the constraint is changed to a larger value by adding 0.01 to the constraint value as in (2.2) in order to extend the feasible region.

$$X \leq ub \Rightarrow X \leq ub + 0.01 \tag{2.2}$$

If the largest Lagrange multiplier is associated with one of the active nonlinear inequality constraints and if it is a minimum constraint, then it is subtracted by 0.01 (see (2.3)), or if it is a maximum constraint, then it is added by 0.01 (see (2.4)) and or if the constraint is neither, then the constraint is not modified as it represents a nonnegative condition.

$$g_i(X) \leq 0 \Rightarrow g_i(X) + 0.01 \leq 0 \tag{2.3}$$

$$g_i(X) \leq 0 \Rightarrow g_i(X) - 0.01 \leq 0 \tag{2.4}$$

The MATLAB file for the nonlinear inequality constraints was provided by ABB and the modification of it can be found in Appendix A.
When the feasible set is extended by altering the active constraint, the optimization problem is solved by the *fmincon* solver again starting from the previous local solution. The decrease of the optimal objective function due to the modification is analyzed and the whole process is repeated until there is no decrease in the optimal objective function. An example is given in Figure 4.

2.4.2 Modification of the Equality Constraints

In addition to the nonlinear inequality constraints, the impact of the nonlinear equality constraints on the optimal objective function is analyzed by converting them into a pair of inequality constraints. The substitution of a pair of inequality constraints into the equality constraint is done by following (2.5)

$$h_j(X) = 0 \Rightarrow -0.05 \leq h_j(X) \leq 0.05 \tag{2.5}$$

An example is given in Figure 5.
Moreover, the sensitivity analysis for these new nonlinear inequality constraints is done by following the perturbation procedure in the previous section. The only difference is that the sensitivity analysis is done on the boundaries of the area instead of on the curve.

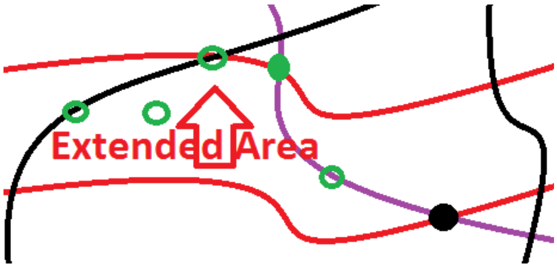


Figure 4: An example of the modification of an inequality constraint for sensitivity analysis. The black curves are inactive whereas both the red curve and the purple curve are active, however the red one has the largest Lagrange multiplier. All of these curves can be any kind of constraints except equality constraints as long as the points are are feasible with respect to the lower and upper bound constraints. Assume that the initial point is led to a feasible solution, which is at the black point. Since it has two active constraints at the local solution, the one which has the largest Lagrange multiplier, which is red in this case, is chosen and given some value (i.e. adding or subtraction) so that the feasible set is extended. When the new feasible region is acquired, the optimization problem is run again starting from where the black point is and the new position is most likely to be the filled green dot because of the active constraint associated with the largest Lagrange multiplier not decreasing much, whereas it could also be one of these green empty dots. The objective function at that point is checked and if it has a smaller value than the previous one and there are active constraints again, the process is repeated until there is no more decrease in the optimal objective function.

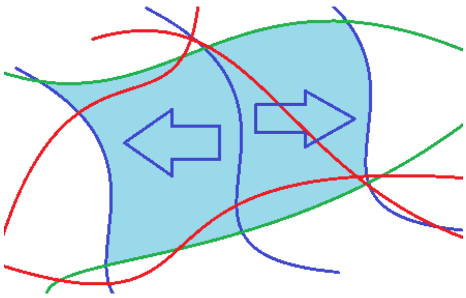


Figure 5: An example of an equality constraint becoming a pair of inequality constraints. The green curves are upper and lower bound constraints, the red curves are nonlinear inequality constraints (referring to one of the two areas separated by the curve in the plane). The blue curve in the middle becomes an area by being converted into two inequality constraints in the both directions.

2.5 Other Model and Method Parameters

We are given two different objective functions, which are called *obj63* and *obj71*. The *obj63* is the problem of minimization of the specific power (the power divided by the production speed) and the *obj71* is for the problem of minimization of the grain size.

The *MultiStart* solver starts with 50 different initial points in attempt to generate the set of the initial points that lead to a feasible solution. After running *fmincon*, the initial points that lead to a feasible solution are kept and and used further.

Furthermore, there are certain tolerances in the numerical optimization for the *fmincon* solver, which are 10^{-6} for first-order optimality condition satisfaction and 10^{-12} for the constraint violation. These also depend on the algorithm used in the solver.

Note that the Lagrange multipliers can be acquired by using the *fmincon* solver alone, i.e. the *MultiStart* solver does not provide the Lagrange multipliers.

The MATLAB files are mexed in order to reduce the execution time. The model is tested on computers with Solaris x86_64 and the version for MATLAB is 2012b (or Version 8.0).

3 Results

3.1 Sensitivity Analysis for *obj71*

obj71 is an objective function describing grain size occurred in the model. It minimizes the grain size subject to the given constraints. It is tested with two different local algorithms in *fmincon* solver.

3.1.1 Active-Set

The optimization problem has 33 upper bound constraints, 33 lower bound constraints, 130 nonlinear inequality constraints, and 10 nonlinear equality constraints. The problem is solved using the local algorithm *active-set* in the *fmincon* solver. The *MultiStart* solver starts with 50 different initial points satisfying upper and lower bound constraints but not satisfying many nonlinear inequality constraints and especially the nonlinear equality constraints. Table 1 shows the values of the variable vector for the local solution.
Table 2 shows that 37 initial points from 50 initial points lead to a feasible solution. There are two nonlinear inequality constraints, which are 108 and 110, not satisfying the negative conditions,i.e. $g_{108}(X) > 0$ and $g_{110}(X) > 0$.
The sensitivity analysis is performed by finding the largest Lagrange multipliers in the active set and modifying the corresponding constraint. Figure 6 and 7 complement each other to show which constraint has the largest Lagrange multiplier. Step in the figures shows how many times the constraint associated with the largest Lagrange multiplier is modified and found a new optimal solution. Figure 6 illustrates which active set group, such as upper bound constraints (U), lower bound constraints (L) and nonlinear inequality constraints (NLI), and constraint has the largest Lagrange multiplier with its value at each step and thus the modified constraint. Figure 7 shows the impact of the modification

Table 1: Variables at the local solution for *obj71AS0*

X_1	0.9764	X_{12}	9.1413e-23	X_{23}	0.96291
X_2	7.3656e-23	X_{13}	1.3073	X_{24}	0.98263
X_3	1.53	X_{14}	1.5351	X_{25}	0.99501
X_4	1.53	X_{15}	1.0236	X_{26}	1.0088
X_5	1.53	X_{16}	1.1429	X_{27}	0.99578
X_6	1.53	X_{17}	1.0769	X_{28}	1.0115
X_7	1.53	X_{18}	0.91649	X_{29}	1.0086
X_8	1.53	X_{19}	1.0008	X_{30}	1.0118
X_9	1.53	X_{20}	1.0178	X_{31}	1.0057
X_{10}	1.53	X_{21}	1.0001	X_{32}	0.9907
X_{11}	-0.16438	X_{22}	0.95	X_{33}	0.8978

Table 2: Information on the feasible set and the optimal point for *obj71AS0*

Number of local solutions acquired	37
Number of distinct local solutions	1
Optimal Objective Function	0.37792
Exitflag	1
Number of Iterations	26
First Order Optimality	3.6999e-14
Constraint Violation	2.327e-13
Non-satisfied Constraints	108, 110

of the constraint on the optimal objective function. The new minimum objective function value is 0.3621, decreased from 0.3779. Since there is a decrease in the objective function value, a better solution has been found.

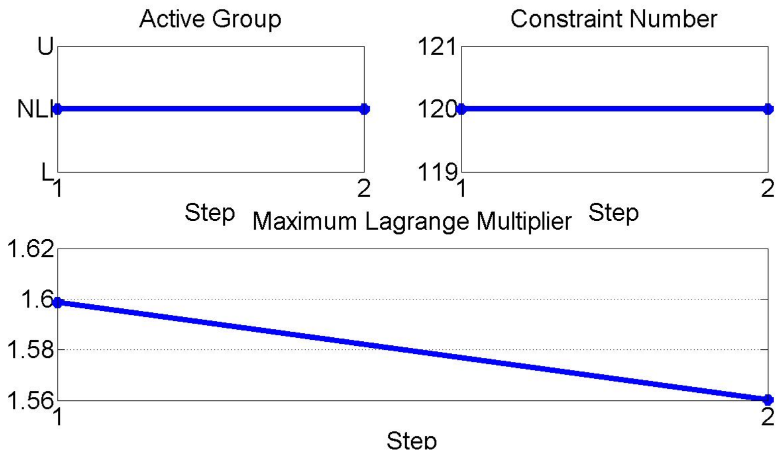


Figure 6: Procedure in *obj71AS0*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

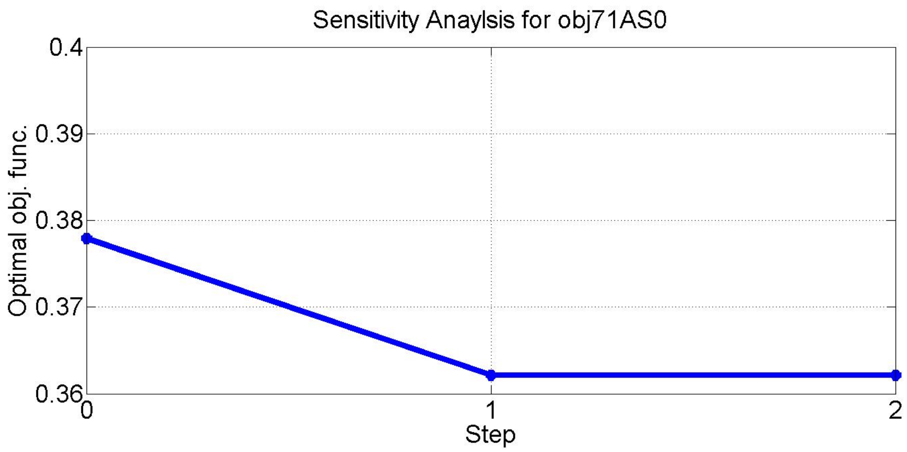


Figure 7: Sensitivity Analysis in *obj71AS0*

Since the nonlinear equality constraints are active by definition, they are converted into a pair of nonlinear inequality constraints. The optimization has now 150 nonlinear inequality constraints with the same upper and lower bound constraints. The problem is solved by the local algorithm *active-set* in the *fmincon* solver again. Table 3 gives the values for the local solution before doing the sensitivity analysis.

Table 4 shows that 38 initial points from 50 initial points lead to a feasible solution generated by *fmincon* solver. There are three nonlinear inequality constraints, which are 53, 108 and 110, not satisfying the negative conditions,i.e.

Table 3: Variables at the local solution for *obj71AS1*

X_1	0.9764	X_{12}	0	X_{23}	0.91391
X_2	0	X_{13}	1.2977	X_{24}	0.93319
X_3	1.53	X_{14}	1.5191	X_{25}	0.94558
X_4	1.53	X_{15}	1.0186	X_{26}	0.95912
X_5	1.53	X_{16}	1.1377	X_{27}	0.94663
X_6	1.53	X_{17}	1.0755	X_{28}	0.96151
X_7	1.53	X_{18}	0.91317	X_{29}	0.95846
X_8	1.53	X_{19}	0.99741	X_{30}	0.96175
X_9	1.53	X_{20}	1.0119	X_{31}	0.95512
X_{10}	1.53	X_{21}	1.0144	X_{32}	0.94191
X_{11}	0.28466	X_{22}	0.95	X_{33}	0.87828

Table 4: Information on the feasible set and the optimal point for *obj71AS1*

Number of local solutions acquired	38
Number of distinct local solutions	1
Optimal Objective Function	0.31373
Exitflag	1
Number of Iterations	28
First Order Optimality	5.79e-14
Constraint Violation	2.2027e-13
Non-satisfied Constraints	53, 108, 110

$g_{53}(X) > 0$, $g_{108}(X) > 0$ and $g_{110}(X) > 0$. The decrease in the optimal objective function by the conversion is 0.06419, which also means that the change in the constraints causes around 16.98% decrease in the optimal value of the objective function.
The sensitivity analysis is applied by finding the largest Lagrange multipliers in the active set and modifying the corresponding constraint. Figure 8 illustrates which active set group and constraint has the largest Lagrange multiplier with its value at each step and thus the modified constraint. Figure 9 shows the impact of the modification of the constraint on the optimal objective function. The new minimum objective function value is 0.3002, decreased from 0.3137.

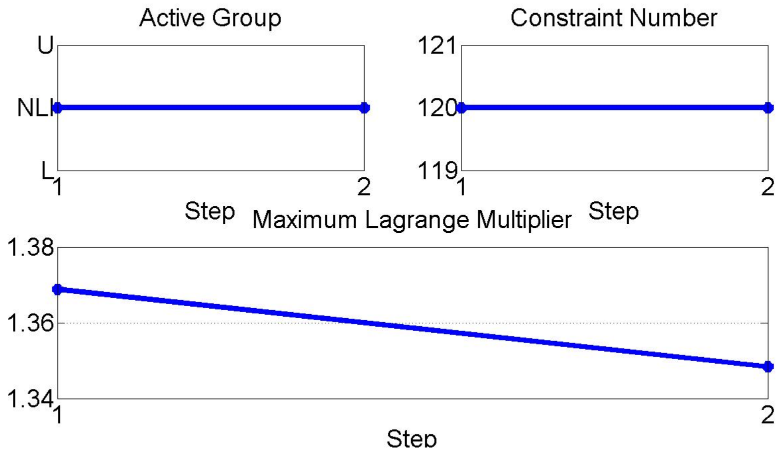


Figure 8: Procedure in *obj71AS1*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

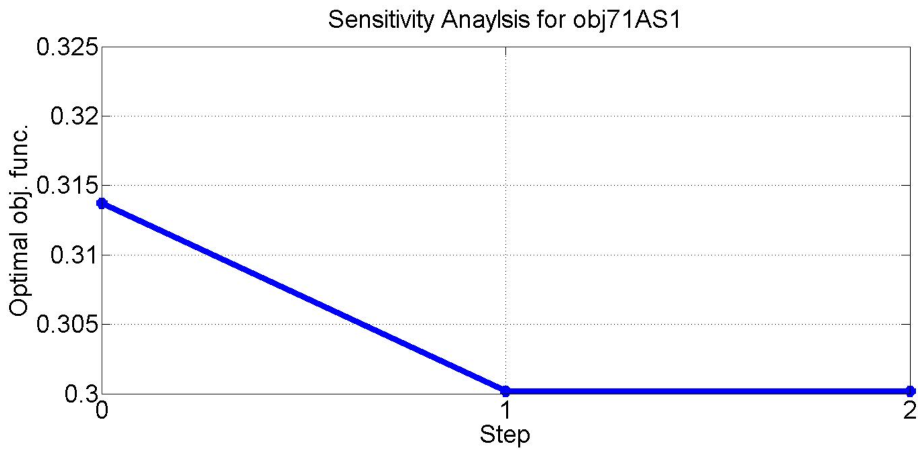


Figure 9: Sensitivity Analysis in *obj71AS1*

3.1.2 SQP

The whole process in the previous section is repeated with the local algorithm *sqp*. Table 5 gives the values for the local solution.

Table 5: Variables at the local solution for *obj71SQP0*

X_1	0.9764	X_{12}	0	X_{23}	0.96291
X_2	0	X_{13}	1.3073	X_{24}	0.98263
X_3	1.53	X_{14}	1.5351	X_{25}	0.99501
X_4	1.53	X_{15}	1.0236	X_{26}	1.0088
X_5	1.53	X_{16}	1.1429	X_{27}	0.99578
X_6	1.53	X_{17}	1.0769	X_{28}	1.0115
X_7	1.53	X_{18}	0.91649	X_{29}	1.0086
X_8	1.53	X_{19}	1.0008	X_{30}	1.0118
X_9	1.53	X_{20}	1.0178	X_{31}	1.0057
X_{10}	1.53	X_{21}	1.0001	X_{32}	0.9907
X_{11}	-0.16438	X_{22}	0.95	X_{33}	0.8978

Table 6: Information on the feasible set and the optimal point for *obj71SQP0*

Number of local solutions acquired	36
Number of distinct local solutions	1
Optimal Objective Function	0.37792
Exitflag	1
Number of Iterations	31
First Order Optimality	1.6155e-06
Constraint Violation	2.0672e-13
Non-satisfied Constraints	56, 108, 110

Table 6 shows that 36 initial points from 50 initial points lead to a feasible solution. There are three nonlinear inequality constraints, which are 56, 108 and 110, not satisfying the negative conditions,i.e. $g_{56}(X) > 0$, $g_{108}(X) > 0$ and $g_{110}(X) > 0$. The sensitivity analysis is performed. The new minimum objective function value is 0.3621, decreased from 0.3779.
The nonlinear equality constraints are converted and Table 7 gives the values for the local solution.
Table 8 shows that 39 initial points from 50 initial points lead to a feasible solution. There are five nonlinear inequality constraints, which are 53, 54, 56, 108 and 110, not satisfying the negative conditions,i.e. $g_{53}(X) > 0$, $g_{54}(X) > 0$, $g_{56}(X) > 0$, $g_{108}(X) > 0$ and $g_{110}(X) > 0$. The decrease in the optimal objective function by the conversion is 0.06419, which also means that the change in the constraints causes around 16.98% decrease in the optimal value of the objective function.
The sensitivity analysis is applied. The new minimum objective function value is 0.3002, decreased from 0.3137.

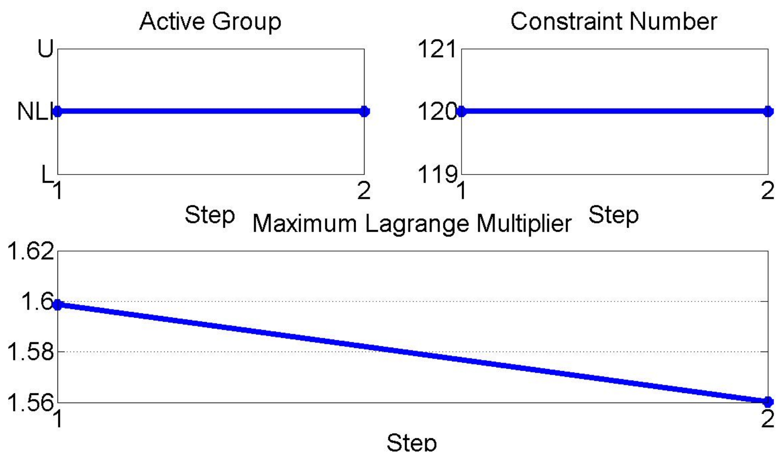


Figure 10: Procedure in *obj71SQP0*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

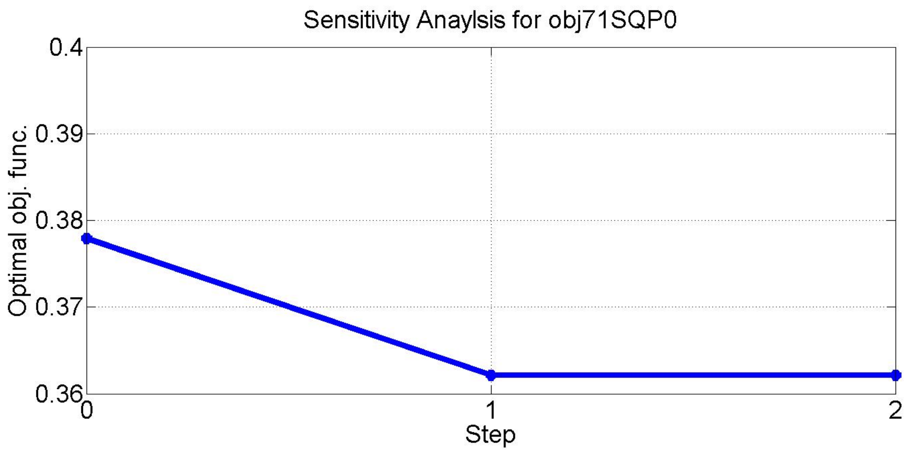


Figure 11: Sensitivity Analysis in *obj71SQP0*

Table 7: Variables at the local solution for *obj71SQP1*

X_1	0.9764	X_{12}	0	X_{23}	0.91391
X_2	0	X_{13}	1.2977	X_{24}	0.93319
X_3	1.53	X_{14}	1.5191	X_{25}	0.94558
X_4	1.53	X_{15}	1.0186	X_{26}	0.95912
X_5	1.53	X_{16}	1.1377	X_{27}	0.94663
X_6	1.53	X_{17}	1.0755	X_{28}	0.96151
X_7	1.53	X_{18}	0.91317	X_{29}	0.95846
X_8	1.53	X_{19}	0.99741	X_{30}	0.96175
X_9	1.53	X_{20}	1.0119	X_{31}	0.95512
X_{10}	1.53	X_{21}	1.0144	X_{32}	0.94191
X_{11}	0.28466	X_{22}	0.95	X_{33}	0.87828

Table 8: Information on the feasible set and the optimal point for *obj71SQP1*

Number of local solutions acquired	39
Number of distinct local solutions	1
Optimal Objective Function	0.31373
Exitflag	1
Number of Iterations	29
First Order Optimality	8.0602e-07
Constraint Violation	2.3093e-14
Non-satisfied Constraints	53, 54, 56, 108, 110

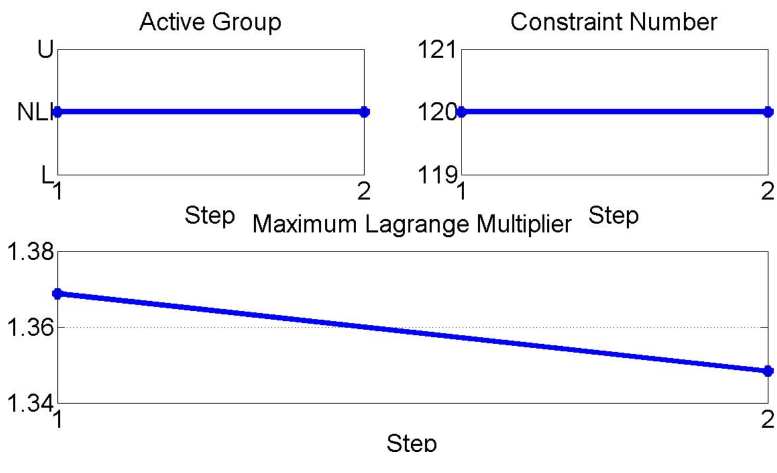


Figure 12: Procedure in *obj71SQP1*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

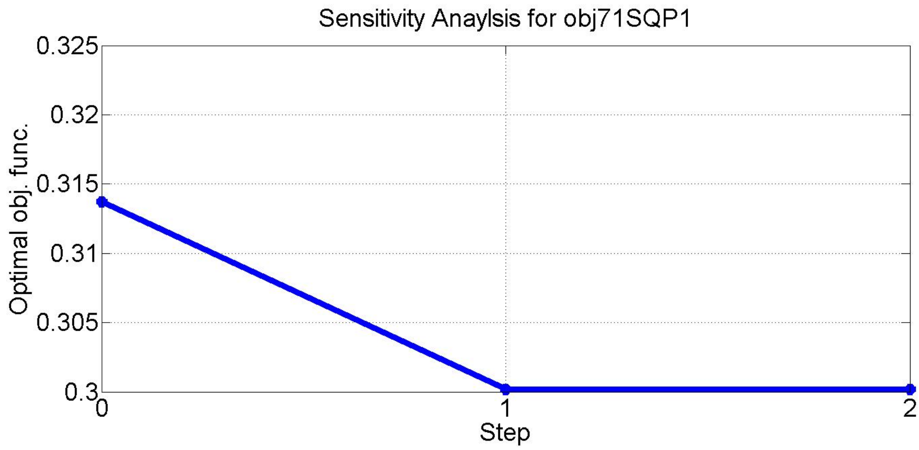


Figure 13: Sensitivity Analysis in *obj71SQP1*

3.2 Sensitivity Analysis for *obj63*

obj63 is an objective function describing specific power in the model. This objective is minimized subject to the given constraints. It is tested with two different local algorithms in *fmincon* solver.

3.2.1 Active-Set

The whole process is again repeated with the objective function *obj63* and the local algorithm *active-set*. Table 9 gives the values for the optimal point in the feasible set.

Table 9: Variables at the optimal point in feasible set for *obj63AS0*

X_1	1.1971	X_{12}	1.7115e-25	X_{23}	1.1976
X_2	1.3834e-22	X_{13}	0.91201	X_{24}	1.2466
X_3	1.53	X_{14}	1.2212	X_{25}	1.2738
X_4	1.53	X_{15}	1.0391	X_{26}	1.274
X_5	1.53	X_{16}	0.48745	X_{27}	1.3414
X_6	1.53	X_{17}	0.6911	X_{28}	1.3746
X_7	1.53	X_{18}	0.2	X_{29}	1.4904
X_8	1.53	X_{19}	0.83048	X_{30}	1.4188
X_9	1.53	X_{20}	0.81881	X_{31}	1.362
X_{10}	1.53	X_{21}	1.2321	X_{32}	1.2603
X_{11}	-0.51001	X_{22}	0.95	X_{33}	1.1429

Table 10 shows that 42 initial points from 50 initial points lead to a feasible solution. However, they give 27 distinct local solutions. Since the values of the objective functions and the variables of these local solutions have similar values to each other, they all are regarded as one single point[4], in which the optimal

[4]To see the difference between different local solutions, please see Appendix B

Table 10: Information on the feasible set and the optimal point for *obj63AS0*

Number of local solutions acquired	42
Number of distinct local solutions	27
Optimal Objective Function	0.60868
Exitflag	1
Number of Iterations	68
First Order Optimality	3.4703e-07
Constraint Violation	5.6188e-13
Non-satisfied Constraints	108, 110

point is chosen. There are two nonlinear inequality constraints, which are 108 and 110, not satisfying the negative conditions,i.e. $g_{108}(X) > 0$ and $g_{110}(X) > 0$ in the optimal point.
The sensitivity analysis is performed. The constraint associated with the largest Lagrange multiplier is changed at step 45 from upper bound constraint number 33 to nonlinear inequality active constraint number 109. The new minimum objective function value is 0.2405, decreased from 0.6087.

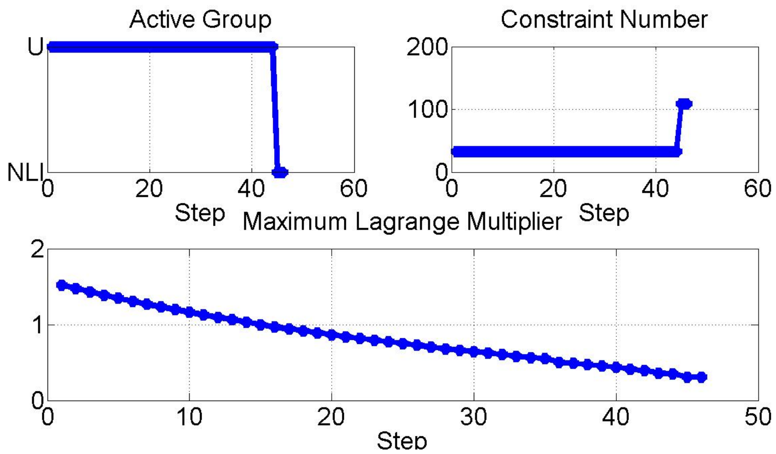


Figure 14: Procedure in *obj63AS0*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

The nonlinear equality constraints are converted and Table 11 gives the values for the optimal point in the feasible set.
Table 12 shows that 41 initial points from 50 initial points lead to a feasible solution. However, they give 32 distinct local solutions. Again, the optimal point is chosen due to the similarities in the values of the objective functions and variables of the local solutions[5]. There are two nonlinear inequality constraints, which are 108 and 110, not satisfying the negative conditions,i.e. $g_{108}(X) > 0$ and $g_{110}(X) > 0$ in the optimal point. The decrease in the optimal objective function by the conversion is 0.02824, which also means that the change in the

[5]To see the difference between different local solutions, please see Appendix B

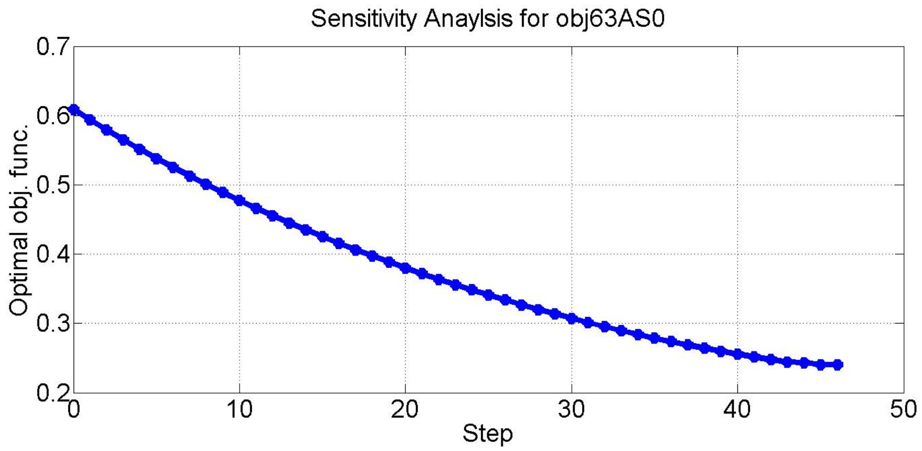


Figure 15: Sensitivity Analysis in *obj63AS0*

Table 11: Variables at the optimal point in feasible set for *obj63AS1*

X_1	1.0418	X_{12}	2.0955e-25	X_{23}	0.98865
X_2	-1.1294e-42	X_{13}	1.0109	X_{24}	1.023
X_3	1.53	X_{14}	1.3713	X_{25}	1.0375
X_4	1.53	X_{15}	1.1114	X_{26}	1.0312
X_5	1.53	X_{16}	0.46998	X_{27}	1.0942
X_6	1.53	X_{17}	0.68489	X_{28}	1.1185
X_7	1.53	X_{18}	0.2	X_{29}	1.2186
X_8	1.53	X_{19}	0.89012	X_{30}	1.1466
X_9	1.53	X_{20}	0.74349	X_{31}	1.1145
X_{10}	1.53	X_{21}	1.2321	X_{32}	1.0442
X_{11}	-0.51001	X_{22}	0.95	X_{33}	1.1429

Table 12: Information on the feasible set and the optimal point for *obj63AS1*

Number of local solutions acquired	41
Number of distinct local solutions	32
Optimal Objective Function	0.58044
Exitflag	1
Number of Iterations	78
First Order Optimality	6.8985e-07
Constraint Violation	4.4605e-13
Non-satisfied Constraints	108, 110

constraints causes around 4.64% decrease in the optimal value of the objective function.
The sensitivity analysis is applied. The constraint associated with the largest Lagrange multiplier is changed at step 43 from upper bound constraint number 33 to nonlinear inequality active constraint number 109. The new minimum objective function value is 0.2405, decreased from 0.5804.

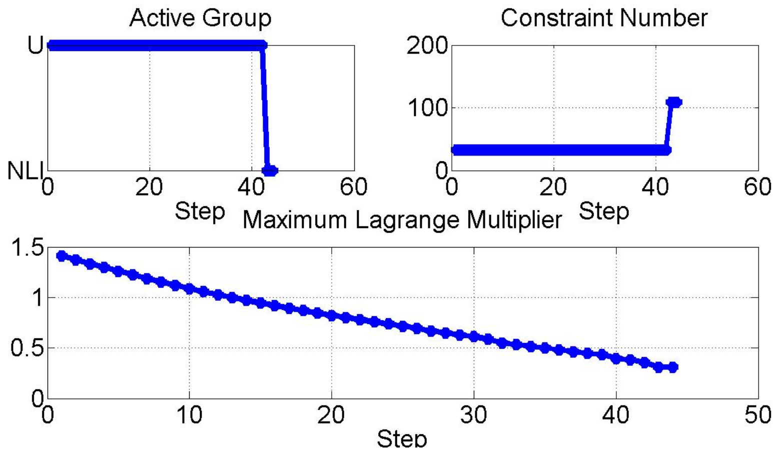


Figure 16: Procedure in *obj63AS1*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

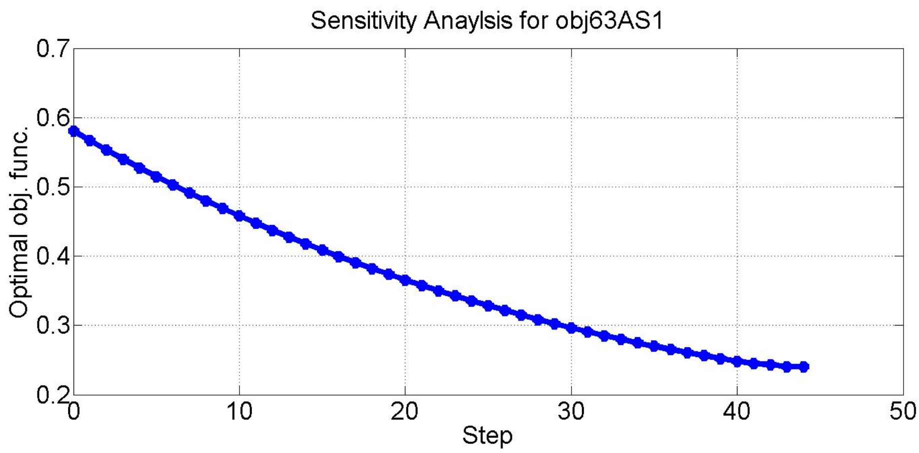


Figure 17: Sensitivity Analysis in *obj63AS1*

3.2.2 SQP

The whole process in the previous section is repeated with the local algorithm *sqp*. Table 13 gives the values for the optimal point in the feasible set.
Table 14 shows that 30 initial points from 50 initial points lead to a feasible solution. However, they give 21 distinct local solutions. Again, the optimal

Table 13: Variables at the optimal point in feasible set for *obj63SQP0*

X_1	1.1971	X_{12}	0	X_{23}	1.1976
X_2	0	X_{13}	0.91201	X_{24}	1.2466
X_3	1.53	X_{14}	1.2212	X_{25}	1.2738
X_4	1.53	X_{15}	1.0391	X_{26}	1.274
X_5	1.53	X_{16}	0.48744	X_{27}	1.3414
X_6	1.53	X_{17}	0.6911	X_{28}	1.3746
X_7	1.53	X_{18}	0.2	X_{29}	1.4904
X_8	1.53	X_{19}	0.83048	X_{30}	1.4188
X_9	1.53	X_{20}	0.81881	X_{31}	1.362
X_{10}	1.53	X_{21}	1.2321	X_{32}	1.2603
X_{11}	-0.51001	X_{22}	0.95	X_{33}	1.1429

Table 14: Information on the feasible set and the optimal point for *obj63SQP0*

Number of local solutions acquired	30
Number of distinct local solutions	21
Optimal Objective Function	0.60868
Exitflag	1
Number of Iterations	82
First Order Optimality	8.3684e-07
Constraint Violation	8.1943e-13
Non-satisfied Constraints	108, 110

point is chosen due to the similarities in the values of the objective functions and variables of the local solutions[6]. There are two nonlinear inequality constraints, which are 108 and 110, not satisfying the negative conditions,i.e. $g_{108}(X) > 0$ and $g_{110}(X) > 0$ in the optimal point.
The sensitivity analysis is performed. The constraint associated with the largest Lagrange multiplier is changed at step 45 from upper bound constraint number 33 to nonlinear inequality active constraint number 109. The new minimum objective function value is 0.2405, decreased from 0.6087.

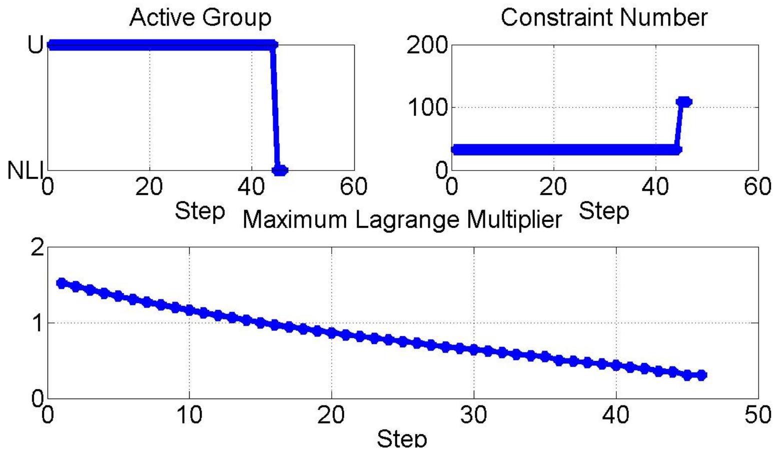


Figure 18: Procedure in *obj63SQP0*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

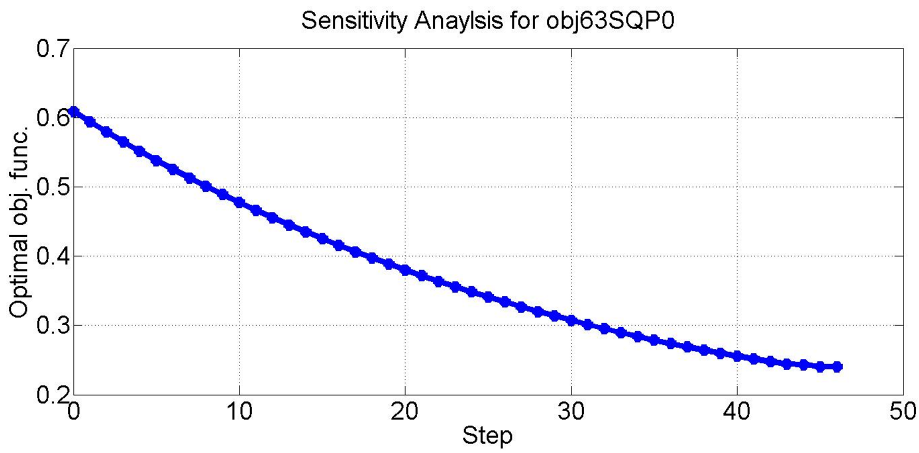


Figure 19: Sensitivity Analysis in *obj63SQP0*

The nonlinear equality constraints are converted and Table 15 gives the values for the optimal point in the feasible set.

[6]To see the difference between different local solutions, please see Appendix B

Table 15: Variables at the optimal point in feasible set for *obj63SQP1*

X_1	1.0418	X_{12}	0	X_{23}	0.98865
X_2	0	X_{13}	1.0109	X_{24}	1.023
X_3	1.53	X_{14}	1.3713	X_{25}	1.0375
X_4	1.53	X_{15}	1.1114	X_{26}	1.0312
X_5	1.53	X_{16}	0.46998	X_{27}	1.0942
X_6	1.53	X_{17}	0.68489	X_{28}	1.1185
X_7	1.53	X_{18}	0.2	X_{29}	1.2186
X_8	1.53	X_{19}	0.89012	X_{30}	1.1466
X_9	1.53	X_{20}	0.74349	X_{31}	1.1145
X_{10}	1.53	X_{21}	1.2321	X_{32}	1.0442
X_{11}	-0.51001	X_{22}	0.95	X_{33}	1.1429

Table 16: Information on the feasible set and the optimal point for *obj63SQP1*

Number of local solutions acquired	33
Number of distinct local solutions	20
Optimal Objective Function	0.58044
Exitflag	1
Number of Iterations	85
First Order Optimality	4.8782e-07
Constraint Violation	1.0933e-13
Non-satisfied Constraints	108, 110

Table 16 shows that 33 initial points from 50 initial points lead to a feasible solution. However, they give 20 distinct local solutions. Again, the optimal point is chosen due to the similarities in the values of the objective functions and variables of the local solutions[7]. There are two nonlinear inequality constraints, which are 108 and 110, not satisfying the negative conditions,i.e. $g_{108}(X) > 0$ and $g_{110}(X) > 0$ in the optimal point. The decrease in the optimal objective function by the conversion is 0.02824, which also means that the change in the constraints causes around 4.64% decrease in the optimal value of the objective function.
The sensitivity analysis is applied. The constraint associated with the largest Lagrange multiplier is changed at step 43 from upper bound constraint number 33 to nonlinear inequality active constraint number 109. The new minimum objective function value is 0.2405, decreased from 0.5804.

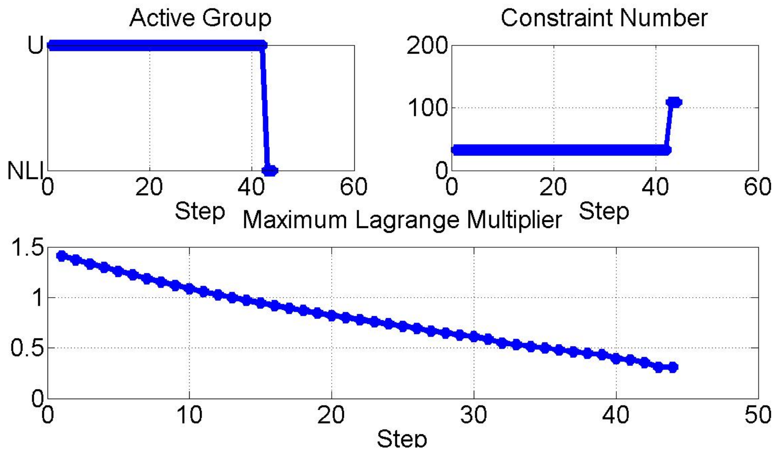


Figure 20: Procedure in *obj63SQP1*. *U* stands for Upper bound constraints, *NLI* stands for Nonlinear inequality constraints and *L* stands for Lower bound constraints. Step shows the number of modification and finding a new optimum.

3.3 Comparison in the Results

In general, both algorithms in the *fmincon* solver give very similar results even though their implementation is different.

All the sensitivity analyses show that there is a better solution available.

The modification of the upper or lower bound constraints has relatively large impact on the objective function function comparing to the modification of the nonlinear inequality constraints.

There are certain constraints that are not satisfied at the optimal solutions. Especially the nonlinear inequality constraint numbers 108 and 110 are not satisfied in all the cases. Nevertheless, when they are applicable, all the nonlinear equality constraints are always satisfied.

The number of iterations or the execution time depends on the complexity of the objective function and the tolerances in numerical optimization.

[7]To see the difference between different local solutions, please see Appendix B

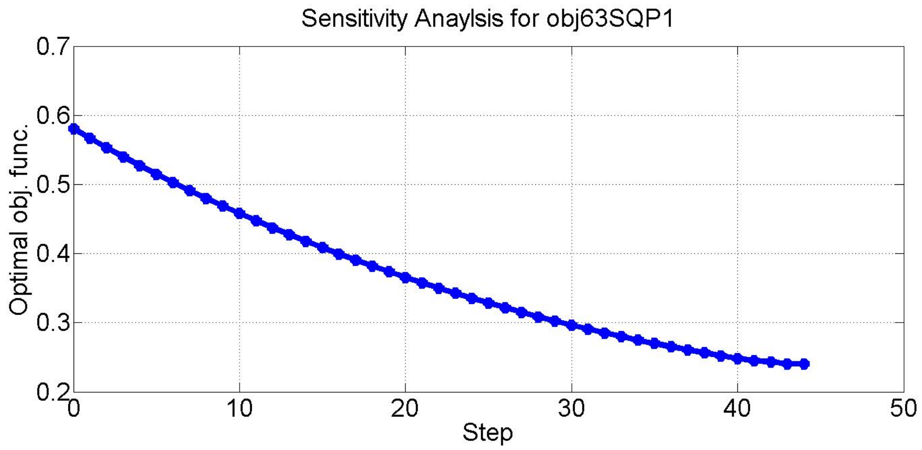


Figure 21: Sensitivity Analysis in *obj63SQP1*

4 Conclusion

4.1 Summary

We briefly summarize the methodology we used when performing the sensitivity analysis. First, 50 different initial points are created by the *MultiStart* solver satisfying the upper and lower bound constraints but they are not feasible with respect to the nonlinear constraints. A feasible set is generated by these points that lead to a feasible solution by using *fmincon* solver and it is kept in order to use in future. The optimal point is also found by the *fmincon* solver and Lagrange multipliers associated with the active constraints at the point are computed by the *fmincon* solver. Based on the information from Lagrange multipliers, the active constraint with the highest Lagrange multiplier is perturbed by increasing or decreasing the right-hand side of inequality constraints and the new problem is resolved. The resolved objective function value is compared to the previous objective function value, the same procedure of computation of all Lagrange multipliers associated with the new active constraints, modification of the constraint with the highest Lagrange multiplier and resolution of the new problem is repeated until there is no more decrease in the objective function. Finally, the whole procedure is repeated again when equality constraints are substituted by the pairs of inequality constraints. The result of this study shows the sensitivity of the given objective functions with different algorithms in *fmincon* solver and their final results.

4.2 Future Research

There are certain variables that cannot change during the optimization problem such as *Variable 2* and *Variable 12*, because their upper and lower bound constraint values are the same. If they are omitted, the optimization problem might become more flexible.

There are certain nonlinear inequality constraints that are not satisfied and they are usually related to the initial points in the feasible set. The initial points or the feasible set could be picked very carefully in order to solve the constraints issue.

References

[1] I. Griva, S. G. Nash, A. Sofer, *Linear and Nonlinear Optimization*, 2nd ed. pp. 491-494, 506 Virginia: SIAM, 2009.

[2] J. Nocedal, S. J. Wright, *Numerical Optimization*, 2nd ed. pp. 321, 341-343 USA: Springer, 2006.

A Modification of Nonlinear Inequality Constraints in MATLAB file

The modification in the three cases, such as increasing a maximum constraint, decreasing a minimum constraint as and a non-changing constraint in MATLAB file.

```
ii_max_ineq=0; %Constraint Number
active %Active Constraint Number

%An example for a minimum constraint
% b2min
for ii=1:N-1
ii_max_ineq=ii_max_ineq+1;

if sum(ii_max_ineq==active)==1
    C(ii_max_ineq)=0.99-OPT.w_exit_OPT(ii)/NXT.b2_min(ii);
else
    C(ii_max_ineq)=1.00-OPT.w_exit_OPT(ii)/NXT.b2_min(ii);
end

% C(ii_max_ineq)=1.00-OPT.w_exit_OPT(ii)/NXT.b2_min(ii); (before)
end

%An example for a maximum constraint
% b2max
for ii=1:N-1
ii_max_ineq=ii_max_ineq+1;

if sum(ii_max_ineq==active)==1
    C(ii_max_ineq)=OPT.w_exit_OPT(ii)/NXT.b2_max(ii)-0.99;
else
    C(ii_max_ineq)=OPT.w_exit_OPT(ii)/NXT.b2_max(ii)-1.00;
end

% C(ii_max_ineq)=OPT.w_exit_OPT(ii)/NXT.b2_max(ii)-1.00; (before)
end

%An example for a non-changing constraint
% tetmin, = 0
for ii=1:N
ii_max_ineq=ii_max_ineq+1;

if sum(ii_max_ineq==active)==1
```

```
    C(ii_max_ineq)=-OPT.tet_OPT(ii);
else
    C(ii_max_ineq)=-OPT.tet_OPT(ii);
end

% C(ii_max_ineq)=-OPT.tet_OPT(ii); (before)
end
```

B Other Distinct Local Solutions

The other distinct local solutions are subtracted by the optimal point in order to see the differences between them. Moreover their number of iterations, first order optimality and constraint violation are compared. The red line in the first optimality and constraint violation plots represents the tolerance used in *fmincon* solver. The order of the figures goes as *obj63SQP0*, *obj63SQP1*, *obj63AS0* and *obj63AS1*, respectively. Note that, since there are exitflags 5 in *obj63AS1*, they give a large difference and if they are omitted, a general difference between the local solutions is around 10^{-11} in the objective function value and around 10^{-5} in the variables. These differences are also valid for the other methods. Since the differences are small, they all can be assumed as one point, which is the optimal point.

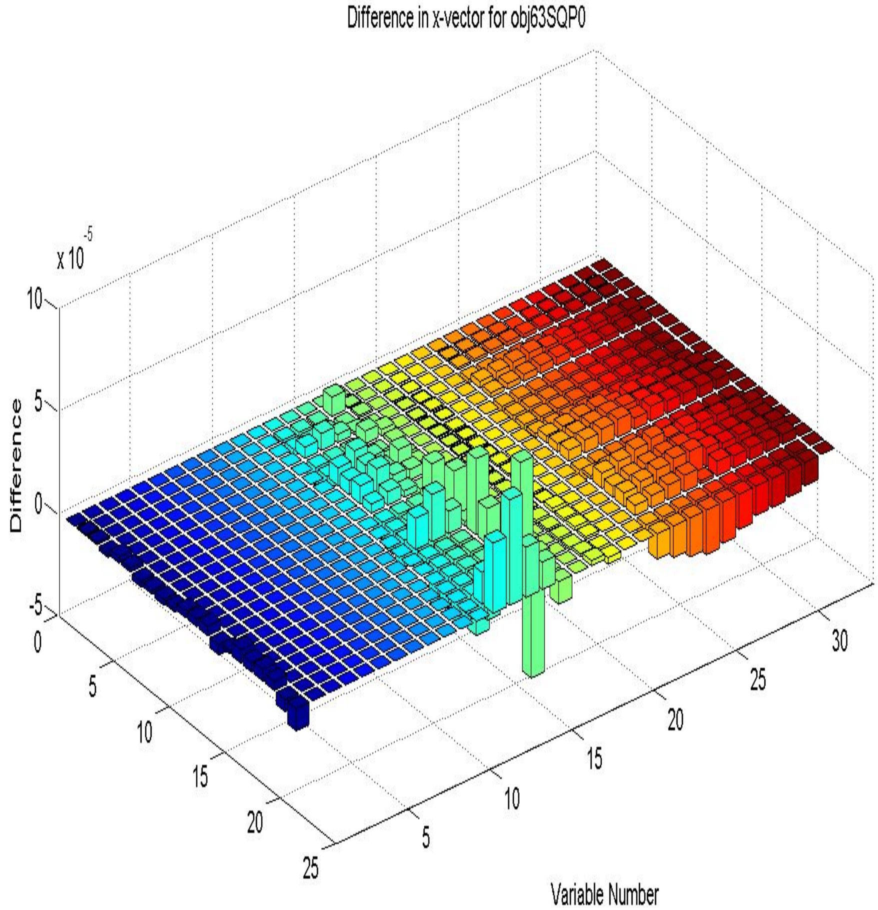


Figure 22: Difference in X-vector for *obj63SQP0*

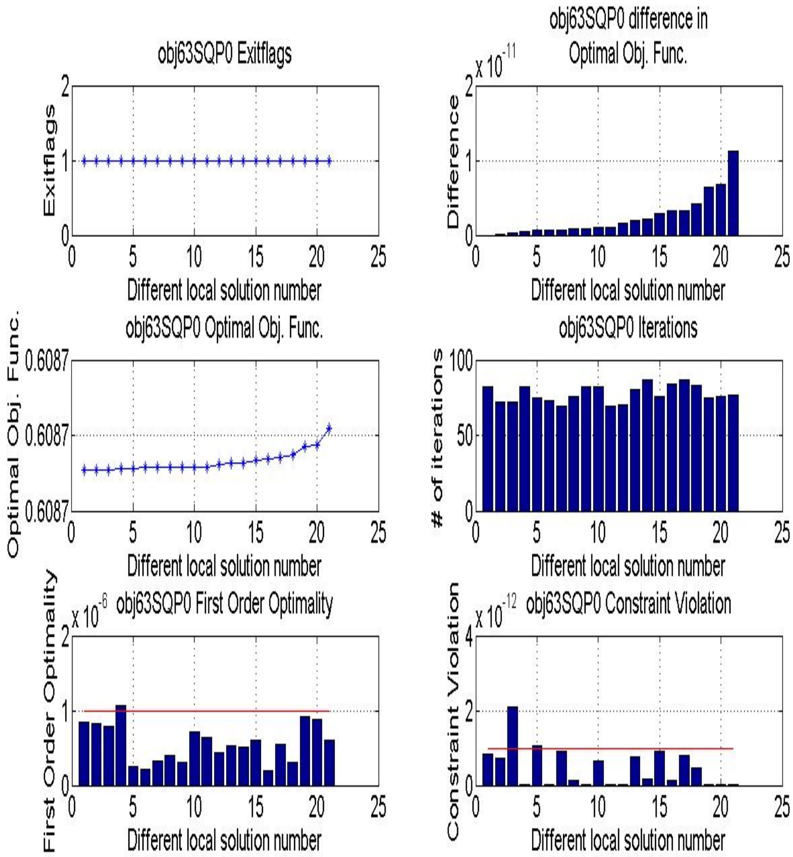


Figure 23: Comparison in local solutions for *obj63SQP0*

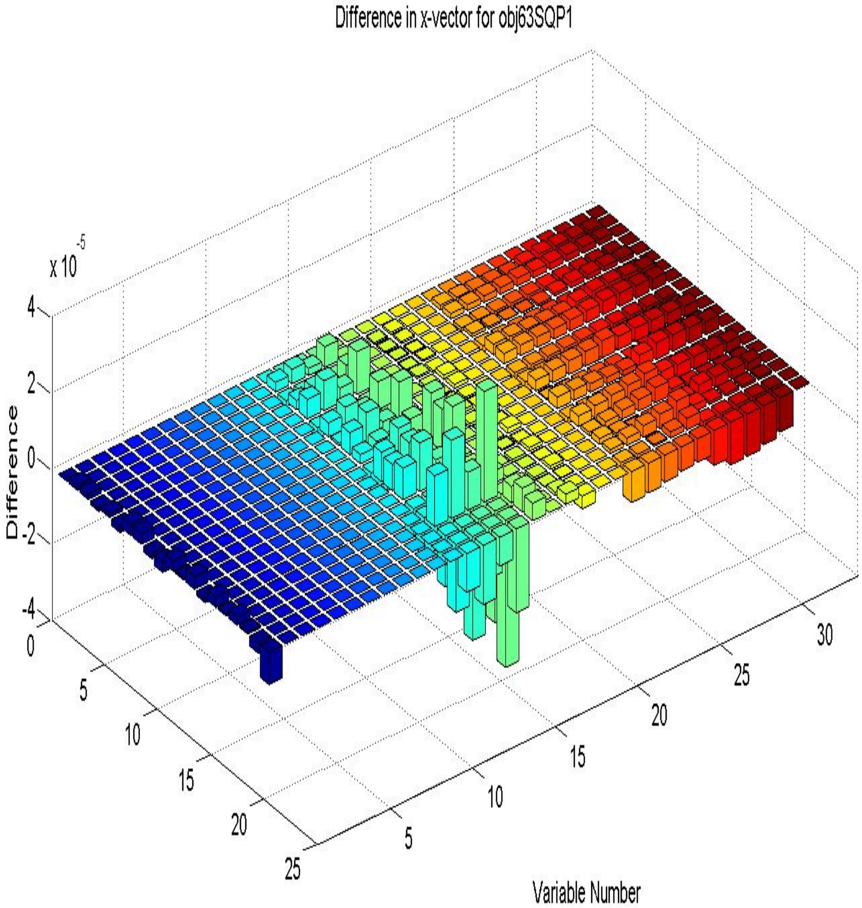


Figure 24: Difference in X-vector for *obj63SQP1*

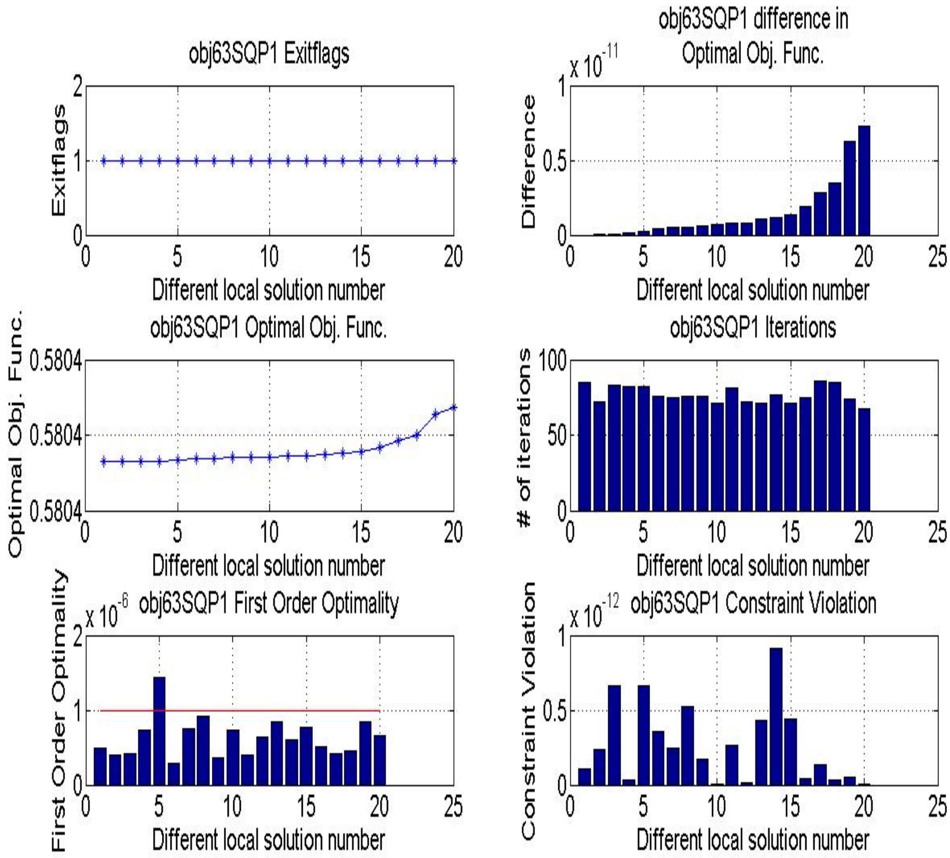


Figure 25: Comparison in local solutions for *obj63SQP1*

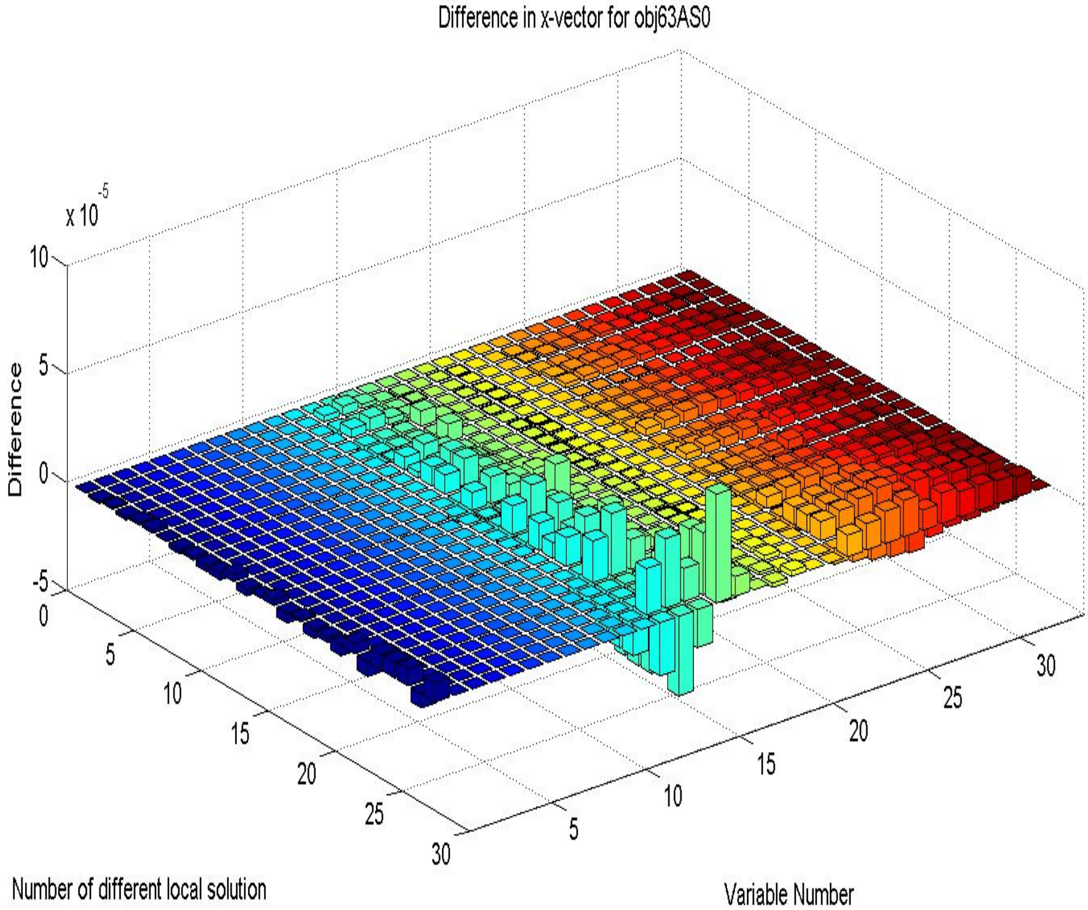


Figure 26: Difference in X-vector for *obj63AS0*

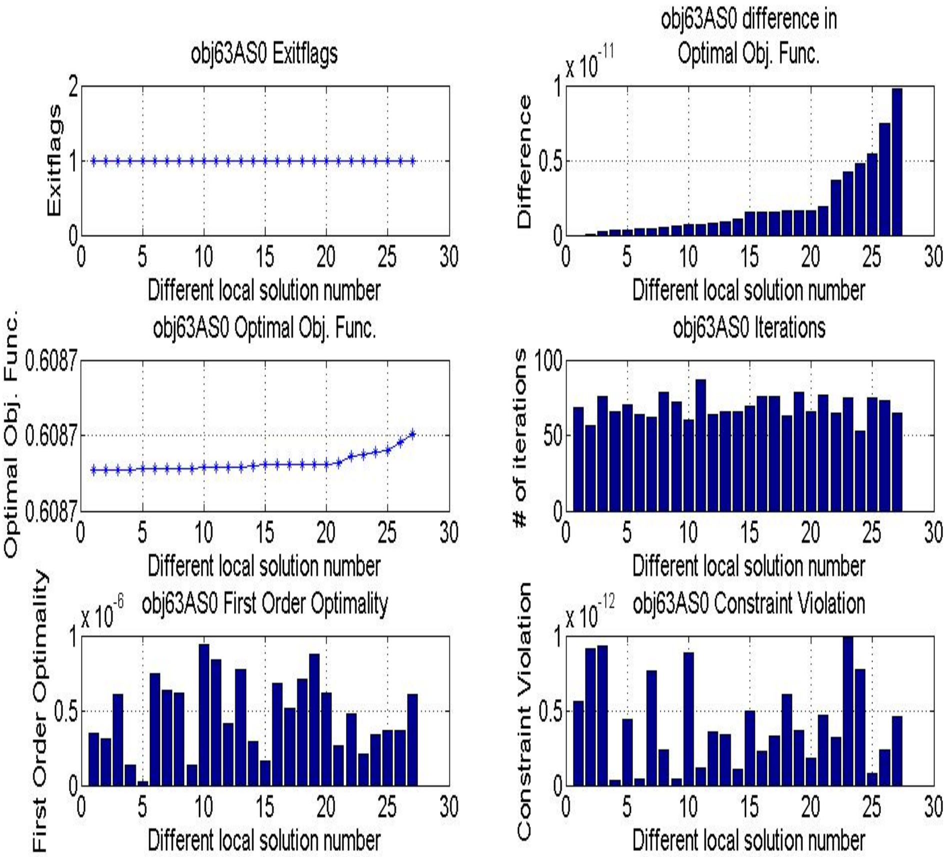


Figure 27: Comparison in local solutions for *obj63AS0*

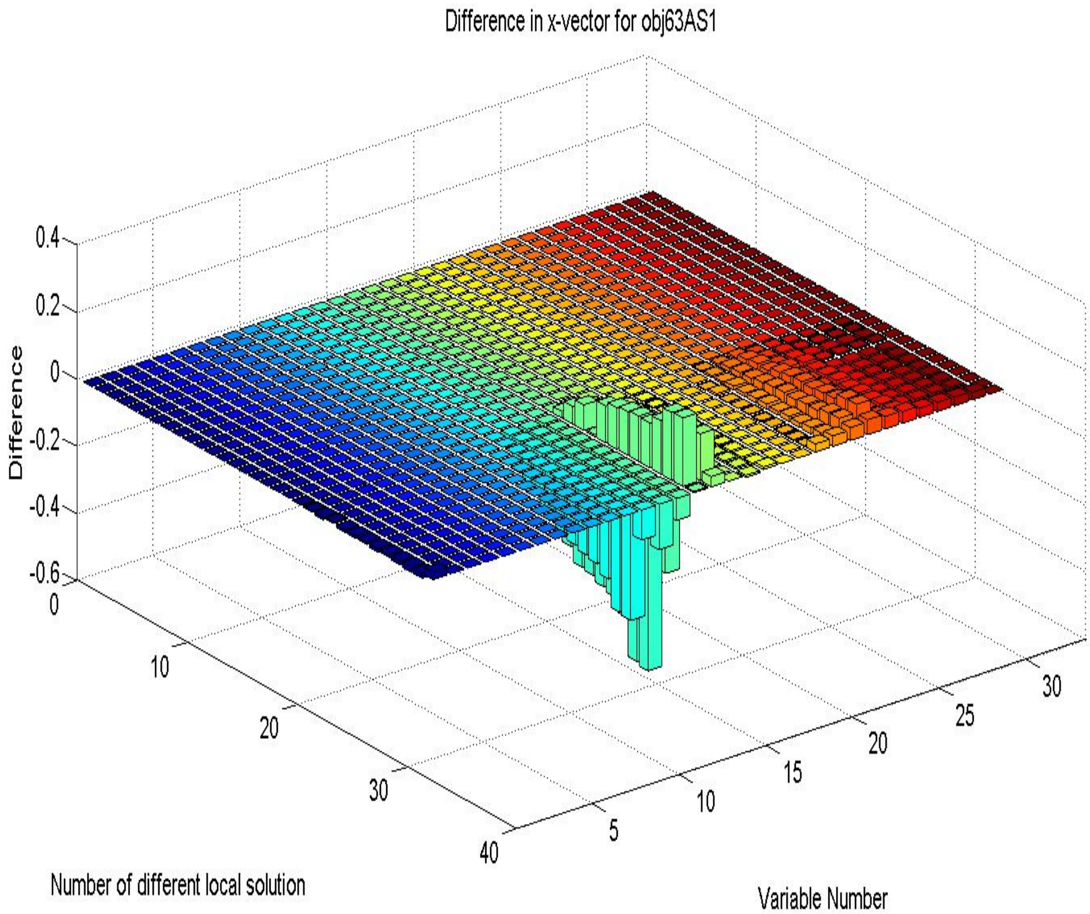


Figure 28: Difference in X-vector for *obj63AS1*

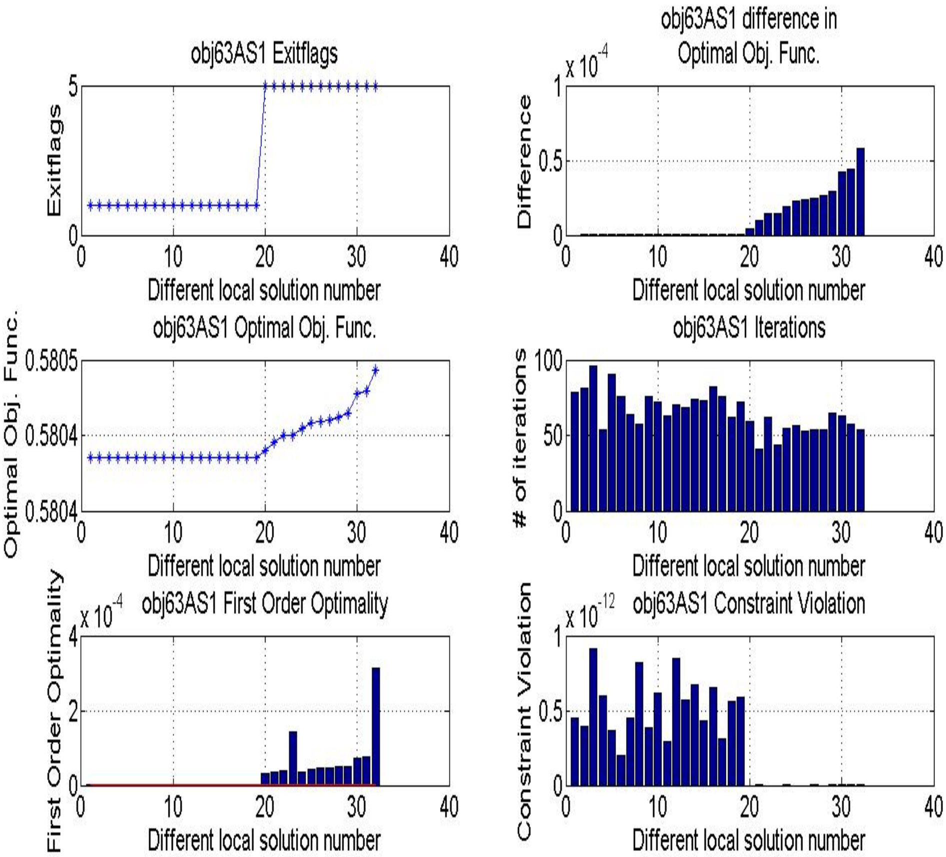


Figure 29: Comparison in local solutions for *obj63AS1*

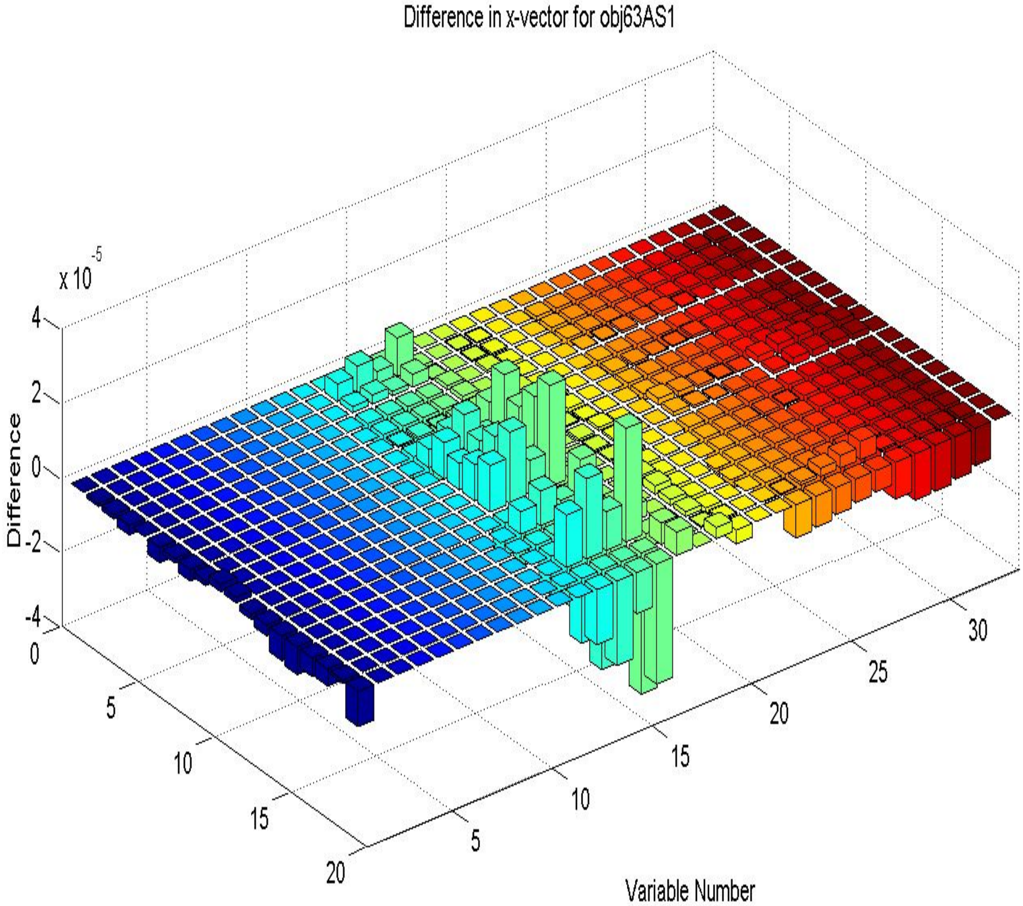


Figure 30: Difference in X-vector for *obj63AS1*

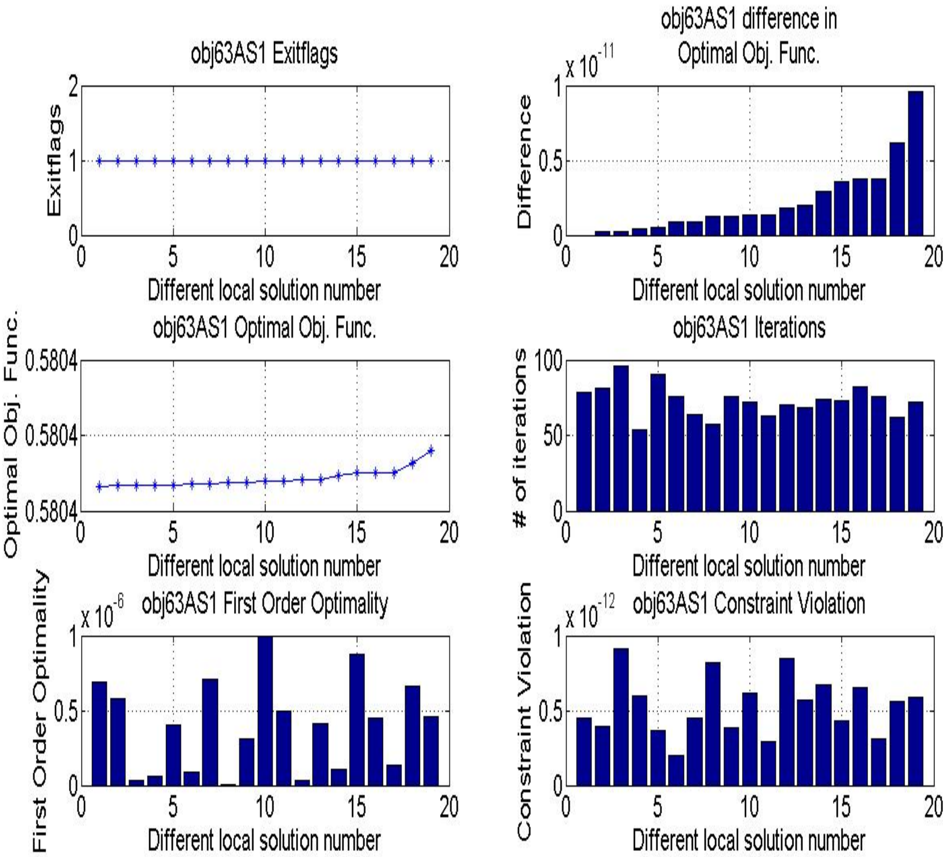


Figure 31: Comparison in local solutions for *obj63AS1*