UPPSALA
UNIVERSITET

Verification of optimality for portfolio consumption problems

Victor Nilsson

Project in Computational Science: Report

January 2014

PROJECT REPORT

Institutionen för informationsteknologi

Abstract

In this report we look at a portfolio consumption optimization problem with a stochastic market price of risk. We go through a previously published proof of optimality in detail and then see how this depends on the assumption of a complete market. Some technical problems are identified and a more general problem of prediction of prices on an incomplete market is addressed. We look at an example model of an incomplete market and recite a theoretical result regarding the ambiguity of prices thereon. To this theoretical result we add a numerical investigation of the impact of this ambiguity in practice. We run Monte-Carlo simulations on a restricted set of reasonable parameter values and interpret the results.

Contents

1 Introduction

A classical problem of interest in quantitative finance is the investors problem, where the task is to find an investment strategy so that the expected utility of the investor is maximal. Many different formalizations of this problem has been studied during the last decades, starting with the work of Merton in [6, 7].

The common way to model the problem is to assume a set of risky assets, usually thought of as stocks, along with a riskless asset, usually thought of as a bond. Then one defines a wealth process that for each time corresponds to the value of the assets owned by the investor at that time. This process is most often required to be self financing, not demanding the investor put more money into it. Lastly comes the tricky question of defining investor utility from the investment. This is usually done via two means. The first is to define a consumption function specifying how much value is extracted from the portfolio, and then defining a function of the consumption, called the running pay-off that is integrated over time to yield investor utility from the consumption. The other approach is applicable in case we are considering the problem with a finite (possibly stochastic) time horizon and define a final pay-off function, sometimes called bequest, giving an investor utility based on the wealth process at the final time. It is also common to combine utility from both consumption and final wealth as we shall see later.

Having formalized the problem as above one usually applies dynamic programming methods from stochastic optimization to get a partial differential equation, called the Hamilton-Jacobi-Bellman (HJB) equation , describing the optimal utility along with the corresponding optimal portfolio and consumption. The important thing to note here is that while it is often straightforward to show that if a portfolio-consumption pair gives optimal expected utility then it must satisfy the HJB equation, showing the opposite can be quite complicated. This is unfortunate since in practice the interesting step is when the investor might get a solution to the equation but would not know wether it actually is optimal or not.

While there are verification theorems for simple market models, it is important from a practical point of view to see if this can also be proved for more realistic models. One such attempt at a realistic model can be found in the article by Honda and Kamimura [3] which we will consider more closely. In [3] they note that empirical studies hava found that the market price of risk is stochastic and time varying, and then proceed to prove verification under certain conditions in a model where the market price of risk is modeled by a linear Gaussian process.

One important assumption in [3] is that the market is complete, i.e. for any price process deriving its value from the assets on the market we can construct a self financing portfolio so they have the same price with probability one. We say that we are replicating the process. Since the price process can be replicated, its value will be determined by the prices of the assets on the market, as these determine the price of the replicating portfolio. In reality this assumption can be questioned as most real markets seem not to be complete. One way to look at completeness is that if a market has more "sources of randomness" than traded assets then the market will be incomplete. It is not difficult to imagine that the number of factors that go into determining the values of assets on a stock market can be larger than the number of assets. However working with incomplete market models puts severe limits on what can be said about the market and the pricing of assets thereon. For a simple introduction to incomplete markets see [2], Chapter 15.

In Section 2 of this report we will take a thorough look into the article [3]. The model is presented in its entirety and the investors problem is formulated. Then the corresponding Hamilton-Jacobi-Bellman equation is derived and simplified to admit a conjectured solution. This explicit solution can then be used to prove a verification theorem. Here we differ from the approach in [3] and the proofs have been slightly reformulated to first prove a general theoretical theorem that is then used to prove two other more applicable results. In Section 3 we look at some of the problems caused if the market is no longer assumed to be complete and Section 4 concludes the report.

2 A dynamic Portfolio-Consumption Problem with stochastic price of risk

2.1 The market model and the investor problem

Assume we have a complete probability space $(\Omega, \mathscr{F}, P)$ and that on this we have a K-dimensional standard Brownian motion $B(t)$, where $t \in [0, T]$. Here we let $\mathscr{F}_t$ be the filtration $\sigma(B(s); 0 < s < t)$, $t \in (0, T)$.

We let the market price of risk be described by the process $X = (X_1, \ldots, X_K)$ that satisfies the stochastic differential equation

$$dX(t) = \mu^X(X(t))dt + \sigma^X dB(t)$$
$$X(0) = x \in \mathbb{R}^K,$$

where $\mu^X(x) = \kappa - Mx$, $\kappa \in \mathbb{R}^K$, $M \in \mathbb{R}^{K \times K}$, and $\sigma^X \in \mathbb{R}^{K \times K}$.

The market will have one riskless asset and K risky assets. Let the price of the riskless asset be denoted by S_0 and have the dynamics

$$dS_0(t) = r(X(t))S_0(t)dt, \quad S_0(0) = 1,$$

where $r : \mathbb{R}^K \to \mathbb{R}$ is given by $r(x) = r_0 + r_1^T x + \frac{1}{2}x^T r_2 x$, $r_1 \in \mathbb{R}$, $r_1 \in \mathbb{R}^K$ and $r_2 \in \mathbb{R}^{K\times K}$ is positive semidefinite.

We denote the price of the risky assets of our market model as $S = (S_1, \ldots, S_K)$ and let S be described by the stochastic differential equations

$$\begin{aligned} &dS_i(t) = S_i(t)\mu_i^S(X(t))dt + S_i(t)\textstyle\sum_{j=1}^K \sigma_{ij}^S(X(t))dB_j(t) \\ &S_i(0) = s_i > 0 \end{aligned}$$

for $i = 1, \ldots, K$. Here $\sigma^S(x) = (\sigma_{ij}^S)_{1\le i,j\le K}$ is a function from $\mathbb{R}^K$ to $\mathbb{R}^{K\times K}$ such that $\Sigma^S(x) := \sigma^S(x)(\sigma^S(x))^T$ is positive definite for all $x \in \mathbb{R}^K$. From the definition of market price of risk we have that $X = \theta(X) := (\sigma^S(X))^{-1}(\mu^S(X) - r(X)\mathbf{1})$, $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^K$, so we get that $\mu^S(x) = r(x)\mathbf{1} + \sigma^S(x)x$, $x \in \mathbb{R}^K$.

Now we want to find an optimal portfolio-consumption strategy for an investor with power-utility. Denote the investors holdings of units of riskless assets at time t by $\eta_0(t)$ and units of the i:th risky asset at time t by η_i. Then the investors wealth $W(t)$ at time t will be

$$W(t) = \sum_{i=0}^K \eta_i(t)S_i(t)$$

and if $c(t)$ is the investors consumption rate at time t the portfolio-consumption pair $(\eta, c) = (\eta_0, \ldots, \eta_K, c)$ will be self-financing if

$$dW(t) = \sum_{i=0}^K \eta_i(t)dS_i(t) - c(t)dt.$$

We also assume there is an initial wealth $w_0 > 0$ such that $W(0) = w_0$. To simplify the equations we look at the fraction of wealth $\phi_i(t)$ invested in the i:th asset at time t, i.e.

$$\phi_i(t) := \begin{cases} \eta_i(t)S_i(t)/W(t), & W(t) \neq 0 \\ 0, & W(t) = 0 \end{cases} \qquad \phi_0(t) := \begin{cases} 1 - \sum_{i=1}^K \phi_i(t), & W(t) \neq 0 \\ 0, & W(t) = 0 \end{cases}$$

Furthermore we define the following classes of stochastic processes

$$\mathcal{L}^1(t_0, t_1) := \left\{ f : \Omega \times [t_0, t_1] \to \mathbb{R} \middle| P\left(\int_{t_0}^{t_1} |f(t)|dt < \infty \right) = 1 \right\}$$

$$\mathcal{L}^2(t_0, t_1) := \left\{ f : \Omega \times [t_0, t_1] \to \mathbb{R}^K \middle| P\left(\int_{t_0}^{t_1} ||f(t)||^2 dt < \infty \right) = 1 \right\}$$

and say that (ϕ, c) is a portfolio-consumption strategy on $[t_0, t_1]$ if $\phi \in \mathcal{L}^2(t_0, t_1)$, $c \in \mathcal{L}(t_0, t_1)$, $c(t) \ge 0$ and $W(t) \ge 0$ $\forall t \in [t_0, t_1]$, where W is the wealth process corresponding to (ϕ, c) in the above way. The set of all portfolio-consumption strategies on $[t_0, t_1]$ is denoted by $\mathcal{H}(t_0, t_1)$.

Using the above formula for ϕ we see that $\phi_0(t) = \eta_0 S_0(t)/W(t)$ and using the dynamics for S the dynamics of W can be written as

$$\begin{aligned}
dW(t) &= \eta_0(r(X(t))S_0(t)dt) - c(t)dt \\
&+ \sum_{i=1}^{K} \eta_i(t) \left(S_i(t)\mu_i^S(X(t))dt + S_i(t) \sum_{j=1}^{K} \sigma_{ij}^S(X(t))dB_j(t) \right) \\
&= W(t)\frac{\eta_0(t)S_0(t)}{W(t)} r(X(t))dt - c(t)dt \\
&+ W(t)\sum_{i=1}^{K} \frac{\eta_i(t)S_i(t)}{W(t)} \mu_i^S(X(t))dt + W(t)\sum_{i=1}^{K}\sum_{j=1}^{K} \frac{\eta_i(t)S_i(t)}{W(t)} \sigma_{ij}^S(X(t))dB_j(t) \\
&= W(t)\left(1 - \phi(t)^T \mathbf{1}\right) r(X(t))dt - c(t)dt \\
&+ \phi(t)^T \mu^S(X(t))dt + W(t)\phi(t)^T \sigma^S(X(t))dB(t) \\
&= W(t)\left[\phi(t)^T(\mu^S(X(t)) - r(X(t))\mathbf{1}) + r(X(t))\right] dt \\
&+ W(t)\phi(t)^T \sigma^S(X(t))dB(t) - c(t)dt.
\end{aligned}$$

To simplify the calculations we introduce $\pi(t) := (\sigma^S(X(t)))^T \phi(t)$ and remember that $\theta(X(t)) = (\sigma^S(X(t)))^{-1}(\mu^S(X(t)) - r(X(t))\mathbf{1})$, since this gives

$$dW(t) = W(t)\left[\pi(t)^T \theta(X(t)) + r(X(t))\right] dt + W(t)\pi(t)^T dB(t) - c(t)dt.$$

Thus instead of (ϕ, c) we will see (π, c) as a portfolio-consumption strategy.

We assumed that our investor had power-utility, i.e. the investors expected utility function is

$$J(t, w, x; \pi, c) = \mathbb{E}^{t,w,x}\left[\int_t^T \frac{c(u)^{1-\gamma}}{1-\gamma} du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right],$$

where we use the notation $\mathbb{E}^{t,w,x}[\,\cdot\,] = \mathbb{E}[\,\cdot\,|W(t) = w, X(t) = x]$. If we let $\mathcal{A}_\gamma(t_0, t_1) \subset \mathcal{H}(t_0, t_1)$ denote the set of admissible strategies, we can define the optimal utility function as

$$V(t, w, x) = \sup_{(\pi,c)\in\mathcal{A}_\gamma(t,T)} J(t, w, x; \pi, c)$$

and then the investors problem is to find an optimal portfolio-consumption strategy (π^*, c^*) such that $V(t, w, x) = J(t, w, x; \pi^*, c^*)$. To find candidates for this we can use dynamic programming principle to find a Hamilton-Jacobi-Bellman equation describing V and (π, c). To do this we first need to find the dynamics for $V(t, W(t), X(t))$ and for this we use Itô's lemma:

Lemma 1 (Itô)**.** *If $Y : \Omega \times [t_0, t_1] \to \mathbb{R}^N$ is an Itô process with dynamics $dY(t) = \mu(t)dt + \sigma(t)dB(t)$ then for a suitably differentiable function $f : [t_0, t_1] \times \mathbb{R}^N \to \mathbb{R}$ there holds*

$$\begin{aligned}
d(f(t, Y(t))) &= \left(f_t(t, Y(t)) + \langle \mu(t), f_x(t, Y(t)) \rangle + \frac{1}{2}\langle \sigma(t)\sigma(t)^T, f_{xx}(t, Y(t)) \rangle \right) dt \\
&+ \langle f_x(t, Y(t)), \sigma(t)dB(t) \rangle.
\end{aligned}$$

Here we use the Frobenius product defined for two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same dimension as $\langle A, B\rangle = \sum_{ij} a_{ij}b_{ij} = \mathrm{tr}[AB^T]$.
To use this for $f = V$ we take as $Y(t) = (W(t), X_1(t), \dots, X_K(t))^T$, and if we let $\mu^W(t) = W(t)\left[\pi(t)^T\theta(X(t)) + r(X(t))\right] - c(t)$ and $\sigma^W(t) = W(t)\pi(t)^T$ we get that $dW(t) = \mu^W(t)dt + \sigma^W(t)dB(t)$ and then $\mu(t) = (\mu^W(t), \mu_1^X(t), \dots, \mu_K^X(t))^T$ and

$$\sigma(t) = \begin{pmatrix} \sigma_1^W(t) & \dots & \sigma_K^W(t) \\ \sigma_{11}^X & \dots & \sigma_{1K}^X \\ \vdots & \ddots & \vdots \\ \sigma_{K1}^X & \dots & \sigma_{KK}^X \end{pmatrix}.$$

This gives us that

$$\begin{aligned} dV &= \Big(V_t + \mu^W(t)V_w + \langle\mu^X(t), V_x\rangle \\ &+ \frac{1}{2}(\sigma^W(t)(\sigma^W(t))^T V_{ww} + \langle\sigma^W(t)(\sigma^X)^T, V_{wx}^T\rangle \\ &+ \langle\sigma^X(\sigma^W(t))^T, V_{wx}\rangle + \langle\sigma^X(\sigma^X)^T, V_{xx}\rangle)\Big)dt \\ &+ V_w\sigma^W(t)dB(t) + \langle V_x, \sigma^X dB(t)\rangle, \end{aligned}$$

where V and all its partial derivatives are evaluated at $(t, W(t), X(t))$. Plugging in the expressions for $\mu^W(t)$ and $\sigma^W(t)$ and setting $\Sigma^X := \sigma^X(\sigma^X)^T$ we get

$$\begin{aligned} dV &= \big(V_t + (W(t)\left[\pi(t)^T\theta(X(t)) + r(X(t))\right] - c(t))V_w + \mu^X(t)^T V_x \\ &+ \frac{1}{2}(W(t)^2||\pi(t)||^2 V_{ww} + W(t)\pi(t)^T(\sigma^X)^T V_{wx} \\ &\quad + W(t)(\sigma^X(\pi(t)^T)^T)^T V_{wx} + \mathrm{tr}[\Sigma^X V_{xx}^T])\big)\, dt \\ &+ W(t)V_w\pi(t)^T dB(t) + V_x^T\sigma^X(dB(t) \\ &= \big(V_t + W(t)\left[\pi(t)^T\theta(X(t)) + r(X(t))\right]V_w - c(t)V_w + \mu^X(t)^T V_x \\ &+ \frac{1}{2}W(t)^2||\pi(t)||^2 V_{ww} + W(t)(\sigma^X\pi(t))^T V_{wx} + 1/2\mathrm{tr}[\Sigma^X V_{xx}^T]\Big)\, dt \\ &+ W(t)V_w\pi(t)^T dB(t) + V_x^T\sigma^X dB(t) \end{aligned}$$

and with this in mind we define the differential operator

$$\begin{aligned} \mathcal{D}^{\pi,c}G(t,w,x) &= G_t + w\left[\pi^T\theta(x) + r(x)\right]G_w - cG_w + \mu^X(t)^T G_x \\ &+ 1/2w^2||\pi||^2 G_{ww} + w(\sigma^X\pi)^T G_{wx} + 1/2\mathrm{tr}[\Sigma^X G_{xx}^T]. \end{aligned}$$

2.2 The Hamilton-Jacobi-Bellman equation

Here we derive a HJB equation for our investor problem through the means of dynamic programming. This is well known method and can be found in many texts on the subject, see e.g. [2], Chapter 19.

Theorem 2. *If there exists an optimal portfolio-consumption pair* (π^*, c^*) *and the corresponding value function* V *is of class* $C^{1,2}$*, then* $G(t,w,x) = V(t,w,x)$ *satisfies the following Hamilton-Jacobi-Bellman equation*

$$\sup_{\pi \in \mathbb{R}^K, c \geq 0} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c} G(t,w,x) \right\} = 0$$

with boundary condition

$$G(T,w,x) = \frac{w^{1-\gamma}}{1-\gamma}.$$

Proof. Let $(t,x) \in [0,T) \times \mathbb{R}^K$ be fixed an chose $h > 0$ such that $t+h < T$. Assume we use an arbitrary portfolio-consumption strategy (π, c) in the time interval $[t, t+h]$ and then switch to the optimal control (π^*, c^*) in $(t+h, T]$. Denote this new concatenated control law as $(\pi * \pi^*, c * c^*)$. Then we get that the expected utility becomes

$$J(t,w,x;\pi * \pi^*, c * c^*) = \mathbb{E}^{t,w,x}\left[\int_t^{t+h} \frac{c(u)^{1-\gamma}}{1-\gamma} du + \int_{t+h}^{T} \frac{c^*(u)^{1-\gamma}}{1-\gamma} du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right]$$

$$= \mathbb{E}^{t,w,x}\left[\int_t^{t+h} \frac{c(u)^{1-\gamma}}{1-\gamma} du + \mathbb{E}^{t+h,W(t+h),X(t+h)}\left[\int_{t+h}^{T} \frac{c^*(u)^{1-\gamma}}{1-\gamma} du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right]\right]$$

$$= \mathbb{E}^{t,w,x}\left[\int_t^{t+h} \frac{c(u)^{1-\gamma}}{1-\gamma} du + V(t+h, W(t+h), X(t+h))\right].$$

Here we can express V with Itô's lemma as

$$\begin{aligned} V(t+h, W(t+h), X(t+h)) &= V(t,w,x) + \int_t^{t+h} \mathcal{D}^{\pi(u),c(u)} V(u, W(u), X(u)) du \\ &+ \int_t^{t+h} \left(W(u) V_w \pi(u)^T + V_x^T \sigma^X\right) dB(u). \end{aligned}$$

Since the first term is deterministic and the last term is an Itô integral, hence a martingale, we get that

$$\begin{aligned} J(t,w,x;\pi * \pi^*, c * c^*) &= V(t,w,x) \\ &+ \mathbb{E}^{t,w,x}\left[\int_t^{t+h} \left(\frac{c(u)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi(u),c(u)} V(u, W(u), X(u))\right) du\right]. \end{aligned}$$

We note that there holds $J(t,w,x;\pi * \pi^*, c * c^*) - V(t,w,x) \leq 0$ with equality for $(\pi, c) = (\pi^*, c^*)$. Hence if we divide by h and take the limit as h goes to zero we obtain that

$$0 \geq \lim_{h \to 0} \mathbb{E}^{t,w,x}\left[\frac{\int_t^{t+h} \left(\frac{c(u)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi(u),c(u)} V(u, W(u), X(u))\right) du}{h}\right]$$

$$= \mathbb{E}^{t,w,x}\left[\frac{c(t)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi(t),c(t)} V(t, W(t), X(t))\right] = \frac{c(t)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi(t),c(t)} V(t,w,x)$$

$$\Rightarrow \frac{c(t)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi(t),c(t)} V(t,w,x) \leq 0$$

again with equality for $(\pi, c) = (\pi^*, c^*)$, proving the HJB equation. For the boundary condition we see that

$$\begin{aligned} V(T,w,x) &= \sup_{(\pi,c)\in\mathcal{A}_\gamma(T,T)} \mathbb{E}^{T,w,x}\left[\int_T^T \frac{c(u)^{1-\gamma}}{1-\gamma}du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right] \\ &= \sup_{\pi\in\mathbb{R}^K, c\geq 0} \mathbb{E}^{T,w,x}\left[\frac{w^{1-\gamma}}{1-\gamma}\right] = \frac{w^{1-\gamma}}{1-\gamma}. \end{aligned}$$

□

The HJB-equation is written as a supremum and so is not immediately useful to us, but since the expression is continuously differentiable in π and c we can write down the first order conditions for a maximum in (π^*, c^*):

$$\nabla_\pi\left(\frac{c^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c}G(t,w,x)\right)\Bigg|_{\pi^*} = w\theta(x)G_w + w^2\pi^* G_{ww} + w(\sigma^X)^T G_{wx} = 0$$

$$\Rightarrow \pi^* = -\frac{G_w}{wG_{ww}}\theta(x) - \frac{(\sigma^X)^T G_{wx}}{wG_{ww}}$$

$$\frac{\partial}{\partial c}\left(\frac{c^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c}G(t,w,x)\right)\Bigg|_{c^*} = (c^*)^{-\gamma} - G_w = 0$$

$$\Rightarrow c^* = G_w^{-1/\gamma}.$$

With these candidates for optimal portfolio-consumption (π^*, c^*) we can write the HJB-equation as

$$\begin{aligned} 0 &= \frac{(c^*)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi^*,c^*}G(t,w,x) \\ &= \frac{G_w^{\frac{\gamma-1}{\gamma}}}{1-\gamma} + G_t - \frac{G_w^2}{G_{ww}}||\theta(x)||^2 - \frac{G_w}{G_{ww}}(\sigma^X\theta(x))^T G_{wx} \\ &+ wr(x)G_w - G_w^{\frac{\gamma-1}{\gamma}} + \mu^X(t)G_x \\ &+ \frac{1}{2}w^2\left(\frac{G_w^2}{w^2G_{ww}^2}||\theta(x)||^2 + 2\frac{G_w\theta(t)^T(\sigma^X)^T G_{wx}}{w^2G_{ww}^2} + \frac{||(\sigma^X)^T G_{wx}||^2}{w^2G_{ww}^2}\right)G_{ww} \\ &- \frac{G_w}{G_{ww}}(\sigma^X\theta(x))^T G_{wx} - \frac{1}{G_{ww}}G_{wx}^T\sigma^X(\sigma^X)^T G_{wx} + \frac{1}{2}\text{tr}[\Sigma^X G_{xx}] \\ &= G_t - \frac{1}{2}\frac{G_w^2}{G_{ww}}||\theta(x)||^2 - \frac{G_w}{G_{ww}}(\sigma^X\theta(x))^T G_{wx} + wr(x)G_w + \mu^X(t)G_x \\ &- \frac{1}{2}\frac{1}{G_{xx}}||(\sigma^X)^t G_{wx}||^2 + \frac{1}{2}\text{tr}[\Sigma^x G_{xx}] + \frac{\gamma}{1-\gamma}G_w^{\frac{\gamma-1}{\gamma}} \end{aligned} \quad (1)$$

The above equation is a non-linear but fortunately there is one more simplification that can be made. The following is a standard result, see e.g. [7].

Lemma 3. *For the value function we have that*

$$V(t,\lambda w,x) = \lambda^{1-\gamma}V(t,w,x)$$

Proof. Let W be a solution to

$$\begin{cases} dW(u) = W(u)[\pi(u)^T\theta(X(u)) + r(X(u))]du + W(u)\pi(u)^T dB(u) - c(u)du \\ W(t) = w. \end{cases}$$

Then we have that $W'(u) := \lambda W(u)$ is a solution to

$$\begin{cases} dW'(u) = W'(u)[\pi(u)^T\theta(X(u)) + r(X(u))]du + W'(u)\pi(u)^T dB(u) - c'(u)du \\ W'(t) = w'. \end{cases}$$

where $c'(u) = \lambda c(u)$ and $w' = \lambda w$. In particular we see that if c is admissible then c' is admissible. Given $\varepsilon > 0$ we can fix t, w, x and choose (π, c) such that

$$V(t, w, x) \leq J(t, w, x; \pi, c) + \varepsilon.$$

Furthermore, we see that

$$\begin{aligned} V(t, \lambda w, x) &\geq J(t, \lambda w, x; \pi, \lambda c) \\ &= J(t, w', x; \pi, c') \\ &= \mathbb{E}^{t,w',x}\left[\int_t^T \frac{c'(u)^{1-\gamma}}{1-\gamma}du + \frac{W'(T)^{1-\gamma}}{1-\gamma}\right] \\ &= \lambda^{1-\gamma}\mathbb{E}^{t,w,x}\left[\int_t^T \frac{c(u)^{1-\gamma}}{1-\gamma}du + \frac{W(T)^{1-\gamma}}{1-\gamma}\right] \\ &= \lambda^{1-\gamma}J(t, w, x; \pi, c) \\ &\geq \lambda^{1-\gamma}(V(t, w, x) - \varepsilon). \end{aligned}$$

Hence, we get

$$V(t, \lambda w, x) \geq \lambda^{1-\gamma}(V(t, w, x) - \varepsilon).$$

Now taking the above inequality and changing λ to λ^{-1} and w to λw we get

$$V(t, w, x) \geq \lambda^{\gamma-1}(V(t, \lambda w, x) - \varepsilon)$$

and the two inequalities taken together give

$$\lambda^{1-\gamma}V(t, w, x) - \lambda^{1-\gamma}\varepsilon \leq V(t, \lambda w, x) \leq \lambda^{1-\gamma}V(t, w, x) + \varepsilon.$$

Since ε can be taken arbitrarily close to zero we get the desired equality. $\square$

Having the above Lemma we can now define a new function

$$f(t, x) := ((1-\gamma)V(t, 1, x))^{1/\gamma},$$

which gives us that

$$V(t, w, x) = \frac{w^{1-\gamma}}{1-\gamma}f(t, x)^{\gamma}.$$

Note that this is well defined because $V(t,w,x) \geq 0$ from the fact that $c(t) \geq 0$ and $W(t) \geq 0$ for all t, w, x.
With this separation in the value function we can let $G(t,w,x) = V(t.w.x)\frac{w^{1-\gamma}}{1-\gamma}f(t,x)^{\gamma}$ in equation (1). We get the partial derivatives

$$G_t = \frac{\gamma}{1-\gamma}\left(\frac{w}{f(t,x)}^{1-\gamma} f_t(t,x)\right) \qquad G_x = \frac{\gamma}{1-\gamma}\left(\frac{w}{f(t,x)}^{1-\gamma} f_x(t,x)\right)$$

$$G_w = \left(\frac{f(t,x)}{w}\right)^{\gamma} \qquad G_{ww} = -\frac{\gamma}{w}\left(\frac{f(t,x)}{w}\right)^{\gamma} \qquad G_{wx} = \frac{\gamma}{f(t,x)}\left(\frac{f(t,x)}{w}\right)^{\gamma} f_x(t,x)$$

$$\begin{aligned} G_{xx} &= \frac{\gamma}{1-\gamma}\left(\frac{f(t,x)}{w}\right)^{1-\gamma} f_{xx}(t,x) - \frac{\gamma}{f(t,x)}\left(\frac{f(t,x)}{w}\right)^{1-\gamma} f_x(t,x)f_x(t,x)^T \\ &= \gamma\left(\frac{f(t,x)}{w}\right)^{1-\gamma}\left(\frac{f_{xx}(t,x)}{1-\gamma} - \frac{f_x(t,x)f_x(t,x)^T}{f(t,x)}\right) \end{aligned}$$

If we let $\Pi = \frac{f(t,x)}{w}$ we can rewrite equation (1) as

$$\begin{aligned} 0 &= \frac{\gamma}{1-\gamma}\Pi^{\gamma-1}f_t - \frac{1}{2}\Pi^{2\gamma}\left(-\frac{w}{\gamma}\right)\Pi^{-\gamma}||\theta(x)||^2 - \Pi^{\gamma}\left(-\frac{w}{\gamma}\right)\Pi^{-\gamma}\theta(x)^T(\sigma^X)^T\frac{\gamma}{f}\Pi^{\gamma}f_x \\ &+ wr(x)\Pi^{\gamma} + \mu^X(x)^T\frac{\gamma}{1-\gamma}\Pi^{\gamma-1}f_x - \frac{1}{2}\left(-\frac{w}{\gamma}\right)\Pi^{-\gamma}\left(\frac{\gamma}{f}\Pi^{\gamma}\right)^2||(\sigma^X)^Tf_x||^2 \\ &+ \frac{1}{2}\gamma\Pi^{\gamma-1}\left(\frac{1}{1-\gamma}\mathrm{tr}[\Sigma^X f_{xx}] - \frac{1}{f}\mathrm{tr}[\sigma^X(\sigma^X)^Tf_xf_x^T]\right) + \frac{\gamma}{1-\gamma}\Pi^{\gamma-1} \end{aligned}$$

and after multiplying by $\frac{1-\gamma}{\gamma}\Pi^{1-\gamma}$ we obtain

$$\begin{aligned} 0 &= f_t + \frac{1}{2}\frac{1-\gamma}{\gamma^2}||\theta(x)||^2 f + \frac{1-\gamma}{\gamma}\theta(x)^T(\sigma^X)^Tf_x \\ &+ \frac{1-\gamma}{\gamma}r(x)f + \mu^X(x)^Tf_x + \frac{1}{2}(1-\gamma)\frac{1}{f}||(\sigma^X)^Tf_x||^2 \\ &+ \frac{1}{2}\mathrm{tr}[\Sigma^X f_{xx}] - \frac{1}{2}(1-\gamma)\frac{1}{f}\mathrm{tr}[\sigma^X(\sigma^X)^Tf_xf_x^T] + 1. \end{aligned}$$

We note that $\mathrm{tr}[\sigma^X(\sigma^X)^Tf_xf_x^T] = f_x^T\sigma^X(\sigma^X)^Tf_x = ||(\sigma^X)^Tf_x||^2$ and rearrange the terms, which leads to

$$\begin{aligned} 0 &= f_t + \left(\frac{1-\gamma}{\gamma}\theta(x)^T(\sigma^X)^T + \mu^X(x)^T\right)f_x + \frac{1}{2}\mathrm{tr}[\Sigma^X f_{xx}] \\ &+ \left(\frac{1-\gamma}{2\gamma^2}||\theta(x)||^2 + \frac{1-\gamma}{\gamma}r(x)\right)f + 1, \end{aligned} \tag{2}$$

and furthermore we get that the boundary condition form the HJB-equation gives that $f(T,x) = 1$. This is a linear PDE and the solution is conjectured to be

$$\begin{aligned} f(t,x) &= \int_t^T \exp\left(\alpha(u) + \beta(u)^Tx + \frac{1}{2}x^T\zeta(u)x\right)du \\ &+ \exp\left(\alpha(t) + \beta(t)^Tx + \frac{1}{2}x^T\zeta(t)x\right) \end{aligned}$$

where $\zeta(t)$ is a symmetric matrix for all t, and the boundary condition then becomes $\alpha(T) = 0$, $\beta(T) = 0$ and $\zeta(T) = 0$. We get the following partial derivatives

$$
\begin{aligned}
f_t(t,x) &= -\exp\left(\alpha(t) + \beta(t)^T x + \frac{1}{2}x^T\zeta(t)x\right) \\
&+ \exp\left(\alpha(t) + \beta(t)^T x + \frac{1}{2}x^T\zeta(t)x\right)\left(\dot\alpha(t) + \dot\beta(t)^T x + \frac{1}{2}x^T\dot\zeta(t)x\right)
\end{aligned}
$$

$$
\begin{aligned}
f_x(t,x) &= \int_t^T \exp\left(\alpha(u) + \beta(u)^T x + \frac{1}{2}x^T\zeta(u)x\right)(\beta(u) + \zeta(u)x)\,du \\
&+ \exp\left(\alpha(t) + \beta(t)^T x + \frac{1}{2}x^T\zeta(t)x\right)(\beta(t) + \zeta(t)x)
\end{aligned}
$$

$$
\begin{aligned}
f_{xx}(t,x) &= \int_t^T \exp\left(\alpha(u) + \beta(u)^T x + \frac{1}{2}x^T\zeta(u)x\right) \\
&\quad \left((\beta(u) + \zeta(u)x)(\beta(u) + \zeta(u)x)^T + \zeta(u)\right)du \\
&+ \exp\left(\alpha(t) + \beta(t)^T x + \frac{1}{2}x^T\zeta(t)x\right)\left((\beta(t) + \zeta(t)x)(\beta(t) + \zeta(t)x)^T + \zeta(t)\right)
\end{aligned}
$$

and for convenience we introduce the notion $E(t,x) = \exp\left(\alpha(t) + \beta(t)^T x + \frac{1}{2}x^T\zeta(t)x\right)$. Substituting this into equation (2) along with $\mu^X(x) = \kappa - Mx$, $\theta(x) = x$ and $r(x) = r_0 + r_1^T x + \frac{1}{2}x^T r_2 x$ gives us

$$
\begin{aligned}
0 &= -E(t,x) + E(t,x)\left(\dot\alpha(t) + \dot\beta(t)^T x + \frac{1}{2}x^T\dot\zeta(t)x\right) \\
&+ \left(\frac{1-\gamma}{\gamma}\sigma^X x + \kappa - Mx\right)^T\left(\int_t^T E(u,x)(\beta(u) + \zeta(u)x)du + E(t,x)(\beta(t) + \zeta(t)x)\right) \\
&+ \left(\frac{1-\gamma}{2\gamma^2}x^T x + \frac{1-\gamma}{\gamma}(r_0 + r_1^T x + \frac{1}{2}x^T r_2 x)\right)\left(\int_t^T E(u,x)du + E(t,x)\right) \\
&+ \frac{1}{2}\mathrm{tr}\Big[\Sigma^X \int_t^T E(u,x)\left(\beta(u)\beta(u)^T + \zeta(u)x\beta(u)^T\right. \\
&\quad \left. + \beta(u)(\zeta(u)x)^T + \zeta(u)xx^T\zeta(u)^T + \zeta(u)\right)du\Big] \\
&+ \frac{1}{2}\mathrm{tr}\left[\Sigma^X E(t,x)\left(\beta(t)\beta(t)^T + \zeta(t)x\beta(t)^T + \beta(t)(\zeta(t)x)^T + \zeta(t)xx^T\zeta(t)^T + \zeta(t)\right)\right] \\
&= E(t,x)\Big(-1 + \dot\alpha(t) + \dot\beta(t)^T x + \frac{1}{2}x^T\dot\zeta(t)x \qquad (3) \\
&+ \kappa^T\beta(t) + \frac{1-\gamma}{\gamma}r_0 + \frac{1}{2}\beta(t)^T\Sigma^X\beta(t) + \frac{1}{2}\mathrm{tr}[\Sigma^X\zeta(t)] \\
&+ \beta(t)^T\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)x + \kappa^T\zeta(t)x + \frac{1-\gamma}{\gamma}r_1^T x + 2(\zeta(t)\Sigma^X\beta(t))^T x \\
&+ x^T\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)\zeta(t)x + \frac{1-\gamma}{2\gamma^2}x^T I_k x + \frac{1-\gamma}{2\gamma}x^T r_2 x + \frac{1}{2}x^T\zeta(t)\Sigma^X\zeta(t)x\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_t^T E(u,x)\left(\kappa^T\beta(u) + \frac{1-\gamma}{\gamma}r_0 + \frac{1}{2}\beta(u)^T\Sigma^X\beta(u) + \frac{1}{2}\mathrm{tr}[\Sigma^X\zeta(t)]\right.\\
&+ \beta(u)^T\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)x + \kappa^T\zeta(u)x + \frac{1-\gamma}{\gamma}r_1^Tx + 2(\zeta(u)\Sigma^X\beta(u))^Tx\\
&+ \left. x^T\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)\zeta(u)x + \frac{1-\gamma}{2\gamma^2}x^TI_Kx + \frac{1-\gamma}{2\gamma}x^Tr_2x + \frac{1}{2}x^T\zeta(u)\Sigma^X\zeta(u)x\right)du,
\end{aligned}
$$

where I_K is the K-dimensional identity matrix. Now by letting

$$
\begin{aligned}
H(t,x) &= E(t,x)\left(\dot{\alpha}(t) + \dot{\beta}(t)^Tx + \frac{1}{2}x^T\dot{\zeta}(t)x\right.\\
&+ \kappa^T\beta(t) + \frac{1-\gamma}{\gamma}r_0 + \frac{1}{2}\beta(t)^T\Sigma^X\beta(t) + \frac{1}{2}\mathrm{tr}[\Sigma^X\zeta(t)]\\
&+ \beta(t)^T\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)x + \kappa^T\zeta(t)x + \frac{1-\gamma}{\gamma}r_1^Tx + 2(\zeta(t)\Sigma^X\beta(t))^Tx\\
&+ x^T\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)\zeta(t)x + \frac{1-\gamma}{2\gamma^2}x^TI_Kx\\
&+ \left.\frac{1-\gamma}{2\gamma}x^Tr_2x + \frac{1}{2}x^T\zeta(t)\Sigma^X\zeta(t)x\right)
\end{aligned}
$$

and differentiating equation (3) with respect to time we simply get that $\frac{\partial}{\partial t}H(t,x) = H(t,x)$, hence $H(t,x) = C(x)e^t$. Unfortunately this equation does not seem to be uniquely solvable, but since we are only looking for one solution we can choose the solution $H(t,x) = 0$. Looking at different "powers" of x in the expression for $H(t,x)$ we see that for $H(t,x) = 0$ to be true we must have that

$$
\dot{\zeta}(t) = -\zeta(t)\Sigma^X\zeta(t) - 2\left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)\zeta(t) - \frac{1-\gamma}{\gamma}\left(\frac{1}{\gamma}I_K + r_2\right) \tag{4}
$$

$$
\dot{\beta}(t) = -\zeta(t)\Sigma^X\beta(t) - \zeta(t)\kappa - \left(\frac{1-\gamma}{\gamma}\sigma^X - M\right)^T\beta(t) - \frac{1-\gamma}{\gamma}r_1 \tag{5}
$$

$$
\dot{\alpha}(t) = -\frac{1}{2}\beta(t)^T\Sigma^X\beta(t) - \beta(t)^T\kappa - \frac{1}{2}\mathrm{tr}[\Sigma^X\zeta(t)] - \frac{1-\gamma}{\gamma}r_0. \tag{6}
$$

Here equation (4) is the well known Riccati equation and we see that if there exists a solution to this equation then the solutions to equations (5) and (6) also exists since these are then linear ordinary differential equations.

The Riccati equation will not always be solvable and for choices of parameters and for a more thorough discussion the reader is referred to [3], where it is also stated that if $\gamma > 1$ then there always exists a solution to equation (4). If the equation is solvable then $\alpha(t)$, $\beta(t)$ and $\zeta(t)$ can be plugged into the formula for $f(t,x)$ yielding the following solution to the HJB equation

$$
G(t,w,x) = \frac{w^{1-\gamma}}{1-\gamma}f(t,x)^\gamma \tag{7}
$$

$$
\begin{aligned}
\pi^*(t) &= -\frac{G_w}{wG_{ww}}\theta(x) - \left.\frac{(\sigma^X)^TG_{wx}}{wG_{ww}}\right|_{w=W^*(t),x=X(t)}\\
&= \frac{1}{\gamma}X(t) + \frac{(\sigma^X)^Tf_x(t,X(t))}{f(t,X(t))} \qquad (8)
\end{aligned}
$$

$$
\begin{aligned}
c^*(t) &= G_w^{-1/\gamma}\Big|_{w=W^*(t),x=X(t)} \\
&= \frac{W^*(t)}{f(t,X(t))}
\end{aligned} \tag{9}
$$

where W^* is the wealth process for the portfolio-consumption pair (π^*, c^*).

2.3 Verification theorems

We will now turn to the question of whether the expressions derived in equations (7), (8) and (9) actually solve the original investors problem. For this we will repeatedly use the following functions

$$
\mathcal{E}(t,g) := \exp\left\{\int_0^t g(u)^T dB(u) - \frac{1}{2}\int_0^t ||g(u)||^2 du\right\}
$$

$$
g^{\pi,c}(s) = \int_t^s \frac{c(u)^{1-\gamma}}{1-\gamma} du + G(s, W(s), X(s)), \quad s \in [t,T]
$$

and

$$
h^{\pi,c}(s) = \left[(1-\gamma)\pi(s) + \gamma\frac{(\sigma^X)^T f_x(s,X(s))}{f(s,X(s))}\right]\frac{G(s,W(s),X(s))}{g^{\pi,c}(s)}.
$$

The proof of the verification theorems will depend on the following result.

Lemma 4. *If $g(t) = \tilde{g}(t, X(t))$, where $\tilde{g} : [0,T]\times\mathbb{R}^K \to \mathbb{R}^K$ satisfies the linear growth condition, i.e. there exists a constant $k > 0$ such that $||\tilde{g}(t,x)|| \leq k(1+||x||)$, then $\mathcal{E}(t,g)$ is a martingale, i.e. $\mathbb{E}^{t,w,x}[\mathcal{E}(T,g)] = \mathcal{E}(t,g)$.*

The above lemma is derived as in Bensoussan [1], Lemma 4.1.1, or Lipster and Shiryaev [5]. The theorems and corrolaries in the remainder of this section were first proved in [3] but are given here in a slightly reformulated way.

Theorem 5 (Verification)**.** *Let*

$$
\mathcal{A}_\gamma(t,T) = \left\{(\pi,c) \in \mathcal{H}(t,T) : \mathcal{E}(t,h^{\pi,c}) \text{ is a martingale}\right\}.
$$

If there exists a solution to (4) on $[0,T]$ then the function G defined by (7) satisfies $G = V$ and (π^*, c^*) defined by (8) and (9) is an optimal portfolio-consumption strategy.

Proof. We begin by showing that (π^*, c^*) is admissible. First we see that $\mathcal{E}(t,h^*)$ is a martingale since

$$
h^*(s) = \left[\frac{1-\gamma}{\gamma}X(s) + \frac{(\sigma^X)^T f_x(s,X(s))}{f(s,X(s))}\right]\frac{G(s,W(s),X(s))}{g^*(s)}
$$

and we know that $||f_x(s,x)|| \leq |f(s,x)|\sup_{t\in[s,T]}(|\beta(t)|+|\zeta(t)|||x||)$ so $||f_x(s,x)/f(s,x)|| \leq C(1+||x||)$ for some $C > 0$. Since we have that $g^*(s) \geq G(s,W(s),X(s))$ we get that $h^*(s)$ satisfies the linear growth condition and from Lemma 4 we get that

If W^* is the wealth process for (π^*, c^*) then we get the desired result.

$$\begin{aligned}
dW^*(t) &= W^*(t)\left[\pi^*(t)^T\theta(X(t)) + r(X(t))\right]dt + W^*(t)\pi^*(t)^T dB(t) - c^*(t)dt \\
&= W^*(t)\left[\pi^*(t)^T\theta(X(t)) + r(X(t)) - \frac{1}{f(t,X(t))}\right]dt + W^*(t)\pi^*(t)^T dB(t).
\end{aligned}$$

We se that

$$\begin{aligned}
W^*(t) &= w_0 \exp\left\{\int_0^t \left(\pi^*(u)^T\theta(X(u)) + r(X(u)) - \frac{1}{f(u,X(u))} - \frac{1}{2}||\pi^*(u)||^2\right)du \right. \\
&\quad \left. + \int_0^t \pi^*(u)^T dB(u)\right\}
\end{aligned}$$

is a solution since Itô's lemma yields the same stochastic differential equation and this has a unique solution. Since both $W^*(t) > 0$ and $c^*(t) > 0$ for all $t \in [0,T]$ we get that (π^*, c^*) admissible.

To prove that $G = V$ we use Itô's formula on $g^{\pi,c}$ in the same way as in the proof of the HJB equation and see that

$$\begin{aligned}
dg^{\pi,c}(s) &= \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c}G(s,W(s),X(s))\right]ds + \left[W(s)G_w\pi(t)^T + G_x^T\sigma^X\right]dB(s) \\
&= \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c}G(s,W(s),X(s))\right]ds \\
&+ G(s,W(s),X(s))\left[(1-\gamma)\pi(s) + \gamma\frac{(\sigma^X)^T f_x(s,X(s))}{f(s,X(s))}\right]^T dB(s) \\
&= \left[\frac{c(s)^{1-\gamma}}{1-\gamma} + \mathcal{D}^{\pi,c}G(s,W(s),X(s))\right]ds + g^{\pi,c}(s)h^{\pi,c}(s)^T dB(s). \qquad (10)
\end{aligned}$$

If we let $g^*(s) := g^{\pi^*,c^*}(s)$ and $h^*(s) := h^{\pi^*,c^*}(s)$ then we get that $dg^*(s) = g^*(s)h^*(s)^T dB(s)$ and

$$\begin{aligned}
g^*(s) &= G(t,w,x)\exp\left\{\int_t^s h^*(u)^T dB(u) - \frac{1}{2}\int_t^s ||h^*(u)||^2 du\right\} \\
&= G(t,w,x)\exp\left\{\int_t^{s'} h^*(u)^T dB(u) - \frac{1}{2}\int_t^{s'} ||h^*(u)||^2 du\right\} \\
&\quad \times \exp\left\{\int_{s'}^s h^*(u)^T dB(u) - \frac{1}{2}\int_{s'}^s ||h^*(u)||^2 du\right\} \\
&= g^*(s')\frac{\mathcal{E}(s,h^*)}{\mathcal{E}(s',h^*)}
\end{aligned}$$

for all $s, s' \in [t,T]$. Using the boundary condition to the HJB equation we have that

$$g^{\pi,c}(T) = \int_0^T \frac{c(u)^{1-\gamma}}{1-\gamma}du + \frac{W(T)^{1-\gamma}}{1-\gamma}.$$

and this together with the assumption that $\mathcal{E}(t, h^*)$ is a martingale, gives us

$$\begin{aligned}
V(t,w,x) &= J(t,w,x;\pi^*,c^*) \\
&= \mathbb{E}^{t,w,x}\left[\int_0^T \frac{c^*(u)^{1-\gamma}}{1-\gamma}du + \frac{W^*(T)^{1-\gamma}}{1-\gamma}\right] \\
&= \mathbb{E}^{t,w,x}\left[g^*(T)\right] \\
&= \mathbb{E}^{t,w,x}\left[g^*(t)\frac{\mathcal{E}(T,h^*)}{\mathcal{E}(t,h^*)}\right] \\
&= G(t,w,x)\frac{\mathbb{E}^{t,w,x}\left[\mathcal{E}(T,h^*)\right]}{\mathcal{E}(t,h^*)} \\
&= G(t,w,x).
\end{aligned}$$

To prove optimality we now use equation (10) together with the HJB equation which gives

$$g^{\pi,c}(s) \leq g^{\pi,c}(t) + \int_t^s g^{\pi,c}(u)h^{\pi,c}(u)^T dB(u)$$

and in the same way as before this gives

$$g^{\pi,c}(T) \leq g^{\pi,c}(t)\frac{\mathcal{E}(T,h^{\pi,c})}{\mathcal{E}(t,h^{\pi,c})}.$$

Therefore

$$\begin{aligned}
J(t,w,x;\pi,c) &= \mathbb{E}^{t,w,x}\left[\int_0^T \frac{c^*(u)^{1-\gamma}}{1-\gamma}du + \frac{W^*(T)^{1-\gamma}}{1-\gamma}\right] \\
&= \mathbb{E}^{t,w,x}\left[g^{\pi,c}(T)\right] \\
&\leq \mathbb{E}^{t,w,x}\left[g^{\pi,c}(t)\frac{\mathcal{E}(T,h^{\pi,c})}{\mathcal{E}(t,h^{\pi,c})}\right] \\
&= G(t,w,x)
\end{aligned}$$

which finishes the proof. $\square$

From this theorem we easily get the following corollary.

Corollary 6. *Let*

$$\mathcal{A}_\gamma(t,T) = \left\{ (\pi,c) \in \mathcal{H}(t,T) : \begin{array}{l} \textit{There is some function } \tilde{\pi}[t_0,t_1] \times \mathbb{R} \to \mathbb{R} \\ \textit{satisfying the linear growth condition} \\ \textit{such that } \pi(t) = \tilde{\pi}(t, X(t)) \end{array} \right\}.$$

If there exists a solution to (4) *on* $[0,T]$ *then the function* G *defined by* (7) *satisfies* $G = V$ *and* (π^*, c^*) *defined by* (8) *and* (9) *is an optimal portfolio-consumption strategy.*

Proof. If $(\pi, c) \in \mathcal{A}_\gamma$ then as before we see that $h^{\pi,c}(s)$ satisfies the linear growth condition according to lemma 4 hence $\mathcal{E}(t, h^{\pi,c})$ is a martingale and we can use theorem 5. $\square$

As noted in [3] the set of admissible strategies can be seen as to restrictive, since there are no practical reasons for demanding that the portfolio grow only linearly in market price of risk. However in the case that the investor is more risk seeking than log-utility investors then the following theorem applies.

Theorem 7 (Verification)**.** *Assume that* $0 < \gamma < 1$ *and let*

$$\mathcal{A}_\gamma(t,T) = \mathcal{H}(t,T).$$

If there exists a solution to (4) *on* $[0,T]$ *then the function* G *defined by* (7) *satisfies* $G = V$ *and* (π^*, c^*) *defined by* (8) *and* (9) *is an optimal portfolio-consumption strategy.*

Proof. It has already been shown in theorem 5 that (π^*, c^*) is admissible and that $G = V$. What remains to be shown is that the strategy is optimal. As before we have that

$$g^{\pi,c}(s) \leq g^{\pi,c}(t) + \int_t^s g^{\pi,c}(u)h^{\pi,c}(u)^T dB(u)$$

for $(\pi, c) \in \mathcal{A}_\gamma(t,T)$ and $s \in [t,T]$. Now we define

$$\Phi(s) := \int_t^s ||g^{\pi,c}(u)h^{\pi,c}(u)||^2 du$$

and

$$\tau_n := T \wedge \inf\{s \in [t,T] : \Phi(s) \geq n\}, \quad n \in \mathbb{N}.$$

We see that $\mathbb{E}[\Phi(s)] < n$ for $s \in [t, \tau_n]$, so $\int_t^s g^{\pi,c}(u)h^{\pi,c}(u)^T dB(u)$ is a martingale for $s \in [t, \tau_n]$. From Doob's optional stopping theorem we get

$$\begin{aligned} \mathbb{E}^{t,w,x}\left[g^{\pi,c}(\tau_n)\right] &\leq \mathbb{E}^{t,w,x}\left[g^{\pi,c}(t)\right] + \mathbb{E}^{t,w,x}\left[\int_t^{\tau_n} g^{\pi,c}(u)h^{\pi,c}(u)^T dB(u)\right] \\ &= G(t,w,x). \end{aligned}$$

Now sincere have that $0 < \gamma < 1$, the utility function is bounded from below and we can use Fatou's lemma as follows. We have that $\lim_{n\to\infty} \tau_n = T$ so

$$\begin{aligned} J(t,w,x;\pi,c) &= \mathbb{E}^{t,w,x}\left[\int_0^T \frac{c(u)^{1-\gamma}}{1-\gamma} du + \frac{W^{1-\gamma}}{1-\gamma}\right] \\ &= \mathbb{E}^{t,w,x}\left[g^{\pi,c}(T)\right] \\ &\leq \liminf_{n\to\infty} \mathbb{E}^{t,w,x}\left[g^{\pi,c}(\tau_n)\right] \\ &\leq G(t,w,x) \end{aligned}$$

for all $(\pi, c) \in \mathcal{A}_\gamma$. ∎

3 Incomplete markets and problems therein

3.1 Incomplete markets model and Hamilton-Jacobi-Bellman equation

Observing the market model in Section 2 and recalling what was said about incomplete markets in Section 1 we see that one way of making the model incomplete is to increase the dimension of the Brownian motion $B(t)$ to some $N > K$. However we immediately see that we can no longer calculate the market price of risk as $\theta = (\sigma^S)^{-1}(\mu^S - r\mathbf{1})$ since now σ^S is no longer invertible. So, instead, we can drop the notion of X as the market price of risk for the time being and just increase the dimension of X to N as well. Then we can still have that $\mu^S(x) = r(x)\mathbf{1} + \sigma^S(x)x$. As a side note we see that if $\Sigma^S(x) = \sigma^S(x)(\sigma^S(x))^T$ is assumed to be positive definite for all $x \in \mathbb{R}^N$ then $\Sigma^S(x)$ is invertible and we get that $x = ((\sigma^S(x))^T\sigma^S(x))^{-1}(\sigma^S(x))^T(\mu^S(x) - r(x)\mathbf{1}) = (\sigma^S(x)\Sigma^S(x)^{-1})^T(\mu^S(x) - r(x)\mathbf{1})$, so X is still something similar to the market price of risk in the complete model.

The rest of the model can be handled similarly as before, until we get to $\pi = (\sigma^S)^T\phi$, where we note that while π has dimension N, ϕ has only dimension K. So we see that not all values of $\pi \in \mathbb{R}^N$ are possible but only those in a K-dimensional subspace $\mathrm{Im}[(\sigma^S(x))^T] \in \mathbb{R}^N$. Now we can question if it is still more convenient to consider π rather than ϕ since this subspace changes depending on X. Either way one will now get that the supremum in the corresponding HJB equation is taken over a $K+1$ dimensional space and the PDE is over $N+1$ spacial dimension and one time dimension. If we then proceed to simplify the equation as before we will only get $K+1$ first order conditions and this will in general be not enough to reduce the HJB equation to a linear equation. As noted in the end of [3], the PDE corresponding to our equation (2) becomes essentially nonlinear, and it seems impossible to derive an explicit solution to such a PDE. As we saw in the proofs of the verification theorems the explicit form of the conjectured solution f was crucial in obtaining an estimate on the candidate optimal portfolio, and without this estimate it will not be possible to prove $G = V$ in the same way as we did for the complete market. For the nonlinear equation it would then be necessary to prove existence of a solution f in another way and then hopefully find a way to prove that $||f_x/f||$ satisfies the linear growth condition. This however would likely demand much more advanced methods than we have presented here and even if successful we don't know under what conditions it would be true.

3.2 Difficulties of pricing in incomplete markets

To further illustrate how difficult it can be to work with incomplete models we look at theorem by Hubalek and Schachermayer in [4] that shows great ambiguity in the standard Black-Scholes pricing methods applied to derivatives on assets that cannot be traded. The method used will be the martingale method of option pricing which is covered in [2] Chapter 10,11 and 12.

We will assume we have a probability space $(\Omega, \mathscr{F}, P)$ with two independent Brownian motions $B, B^{\perp}$. Let T be the time horizon and let $(\mathscr{F})_{0\leq t\leq T}$ be the filtration generated by $(B, B^{\perp})$. Also assume that $\mathscr{F} = \{\emptyset, \Omega\}$ and $\mathscr{F}_T = \mathscr{F}$. Given a constant $\rho \in (0,1)$ we define a new Brownian motion

$$\tilde{B}(t) = \rho B(t) + \sqrt{1-\rho^2} B^{\perp}(t) \quad \text{for } t \in [0, T].$$

Here we see that $\mathrm{Cov}[B(t), \tilde{B}(t)] = \rho t$. Now let $\mu, \tilde{\mu} \in \mathbb{R}$ and $r, \sigma, \tilde{\sigma} > 0$ be fixed and consider the standard Black-Scholes model

$$\begin{aligned} dS_0(t) &= rS_0(t)dt \\ dS_1(t) &= \mu S_1(t)dt + \sigma S_1(t)dB(t). \end{aligned}$$

Then we get that

$$\begin{aligned} S_0(t) &= S_0(0)e^{rt} \\ S_1(t) &= S_1(0)\exp(Y(t)) \text{ where } Y(t) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B(t). \end{aligned}$$

Now we also introduce a third asset that cannot be traded

$$d\tilde{S}(t) = \tilde{\mu}\tilde{S}(t)dt + \tilde{\sigma}\tilde{S}(t)d\tilde{B}(t)$$

that is

$$\tilde{S}(t) = \tilde{S}(0)\exp(\tilde{Y}(t)) \text{ where } \tilde{Y}(t) = \left(\tilde{\mu} - \frac{\tilde{\sigma}^2}{2}\right)t + \tilde{\sigma}\tilde{B}(t).$$

This nontraded asset will serve as the underlying asset for a European call option with price process $C(t)$, i.e. $C(T) = \max(\tilde{S}(T) - K, 0)$ where $K > 0$ is the strike price. The idea now is that since B and $\tilde{B}$ are correlated with constant ρ, close to 1, then it seems sensible to take S_1 as a traded surrogate for $\tilde{S}$. Hopefully then we can approximate the price of C using only the traded asset S_1 and the bond S_0. As we shall see this is not the case though.

First we define

$$\mathcal{M}(S_1) := \left\{ Q : \begin{array}{l} Q \text{ is a probability measure on } \mathscr{F} \text{ equivalent to } P, \\ \text{and } S_1/S_0 \text{ is a } Q-\text{martingale.} \end{array} \right\}.$$

Theorem 8 (Habulek-Schachermayer)**.** *For any* $c_0 \in (0, \infty)$ *there is a probability measure* $Q \in \mathcal{M}(S_1)$ *such that*

$$c_0 = e^{-rT}\mathbb{E}_Q[\max(\tilde{S}(T) - K, 0)].$$

In particular the market $\{(S_0(t)), (S_1(t)), (C(t))\}_{t\in[0,T]}$ *where*

$$C(t) = e^{-r(T-t)}\mathbb{E}_Q[\max(\tilde{S}(T) - K, 0)|\mathscr{F}_t],$$

is arbitrage free.

Proof. We use Girsanov's theorem to find a new probability measure which changes μ to r and $\tilde{\mu}$ to $\tilde{r}$ in the 2-dimensional process

$$\begin{bmatrix} dS_1 \\ d\tilde{S} \end{bmatrix} = \begin{bmatrix} \mu \\ \tilde{\mu} \end{bmatrix} dt + \begin{bmatrix} \sigma & 0 \\ \rho\tilde{\sigma} & \sqrt{1-\rho}\tilde{\sigma} \end{bmatrix} \begin{bmatrix} dB \\ dB^{\perp} \end{bmatrix}.$$

We can then define a new measure $Q_\nu \in \mathcal{S}$ by $dQ_\nu/dP = Z_T$, where

$$Z_T = \exp\left[\lambda B(t) + \lambda^{\perp}B^{\perp}(t) - \frac{\lambda^2 + (\lambda^{\perp})^2}{2}t\right], \quad t \in [0, T]$$

and here we have that

$$\begin{aligned} -\begin{bmatrix} \lambda \\ \lambda^{\perp} \end{bmatrix} &= \begin{bmatrix} \sigma & 0 \\ \rho\tilde{\sigma} & \sqrt{1-\rho}\tilde{\sigma} \end{bmatrix}^{-1} \left(\begin{bmatrix} \mu \\ \tilde{\mu} \end{bmatrix} - \begin{bmatrix} r \\ \tilde{r} \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{\mu-r}{\sigma} \\ \frac{\rho(r-\mu)}{\sigma\sqrt{1-\rho^2}} + \frac{\tilde{\mu}-\tilde{r}}{\tilde{\sigma}\sqrt{1-\rho^2}} \end{bmatrix}. \end{aligned}$$

Thus, we get a new 2-dimensional Brownian motion under Q_ν, namely,

$$\begin{bmatrix} B(t) - \lambda t \\ B^{\perp}(t) - \lambda^{\perp}t \end{bmatrix}$$

and the processes $Y(t)$, $\tilde{Y}(t)$ are given by

$$\begin{aligned} X(t) &= \left(r - \frac{\sigma^2}{2}\right)t + \sigma(B(t) - \lambda t) \\ \tilde{X}(t) &= \left(\tilde{r} - \frac{\tilde{\sigma}^2}{2}\right)t + \tilde{\sigma}\left(\rho(B(t) - \lambda t) + \sqrt{1-\rho^2}(B^{\perp}(t) - \lambda^{\perp}t)\right). \end{aligned}$$

If we let

$$\nu = \tilde{r} - \frac{\tilde{\sigma}^2}{2},$$

we see that the random variable

$$U_T^{(\nu)} = \rho(B(t) - \lambda t) + \sqrt{1-\rho^2}(B^{\perp}(t) - \lambda^{\perp}t)$$

is a normal random variable with mean 0 and variance T under the measure Q_ν. Now we get that

$$\begin{aligned}
E(\nu) &= \mathbb{E}_{Q_\nu}\left[\max(\tilde{S}(T) - K, 0)\right] \\
&= \mathbb{E}_{Q_\nu}\left[\max(\tilde{S}(0)\exp(\nu T + \tilde{\sigma} U_T^{(\nu)}) - K, 0)\right] \\
&= \int_{-\infty}^{\infty} \max(\tilde{S}(0)\exp(\nu T + \tilde{\sigma}\sqrt{T} z) - K, 0)\varphi(z)dz,
\end{aligned}$$

where φ is the probability density function for $\mathcal{N}(0,1)$. With some elementary algebra we can then see that

$$E(\nu) = \int_{z_\nu}^{\infty} \left(\tilde{S}(0)e^{\nu T + \tilde{\sigma}\sqrt{T}z} - K\right)\varphi(z)dz,$$

where

$$z_\nu = \frac{\ln\frac{K}{S_0} - \nu T}{\tilde{\sigma}\sqrt{T}}.$$

Here we see that

$$\lim_{\nu\to-\infty} E(\nu) = 0 \text{ and } \lim_{\nu\to\infty} E(\nu) = \infty.$$

Now taking any $c_0 \in (0,\infty)$ we see there is ν^- and ν_+ such that $E(\nu^-) < c_0 < E(\nu^+)$. Since the function $\nu \to E(\nu)$ is continuous the mean value theorem says that there will be $\nu^* \in (\nu^-, \nu^+)$ such that $E(\nu^*) = c_0$. $\square$

This theorem illustrates a fundamental problem with pricing assets in incomplete markets. Even though we can let S_1 and $\tilde{S}$ be arbitrarily well correlated, as long as they are not the same there will always be a large leeway for the market to price derivatives on $\tilde{S}$.

However the above theorem only tells us about what prices are possible, which is not necessary the same as to say the are likely. If most of the range corresponds to very unlikely prices, it might not be a huge problem in practice. Using the formula

$$E(\nu) = \mathbb{E}_{Q_\nu}\left[\max(\tilde{S}(0)\exp(\nu T + \tilde{\sigma} U_T^{(\nu)}) - K, 0)\right]$$

and restricting to "reasonable" values of $\tilde{r}$ and $\tilde{\sigma}$ we can use Monte-Carlo methods to examine the range of prices generated. Determining a reasonable range is not that simple but here we have chosen $\tilde{r} \in (0.5, 1.5)$ and $\tilde{\sigma} \in (0.1, 1)$. Normalizing the price such that $\tilde{S}(0) = 1$ we can plot the prices for different values on K. In Figure 1 we see the range of prices in the case of the strike price K being half of the starting value of $\tilde{S}$. Here wee see that while the volatility seems to have little impact on the price, there is quite an incline along the $\tilde{r}$-axis. In Figure 2 and 3 we see the prices when K equals $\tilde{S}(0)$ and when K is twice as large as $\tilde{S}(0)$ respectively. First we can see that there is an increasing dependence on the volatility as we take larger K. This is

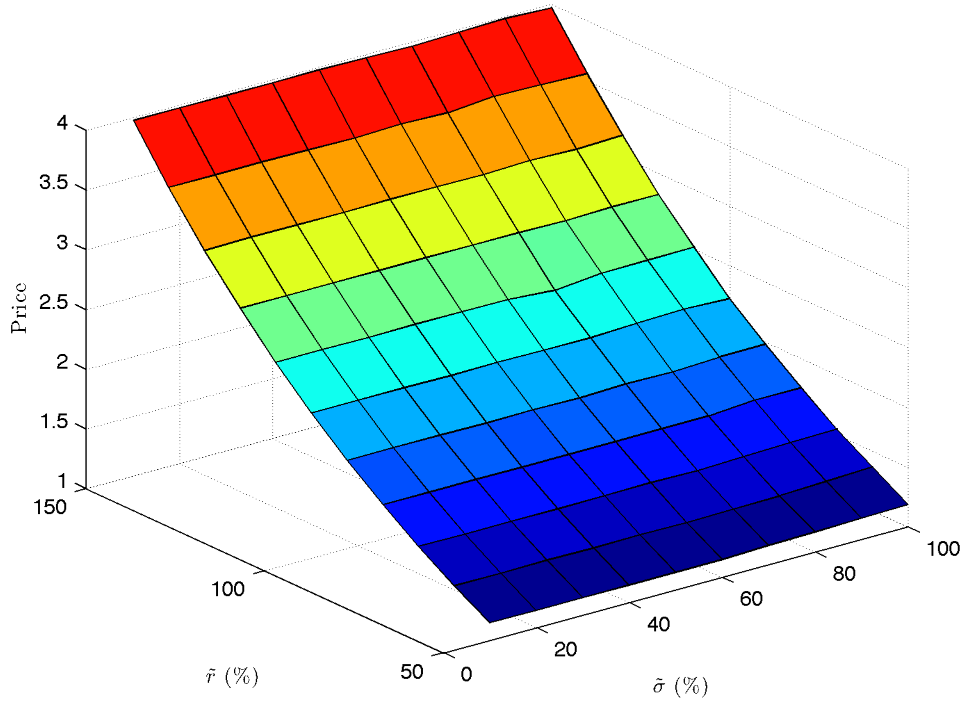

Figure 1: Option prices for $\tilde{S}(0)$ and $K = 0.5$.

natural since if we start out with the $\tilde{S}$ below the strike price, an increased volatility will mean a larger chance to get over the strike price and achieve a non-zero final value on the option. We also see that the price range seems to get slightly narrower with increased strike price, and this is also to be expected since given any realization of $\tilde{S}$,a higher strike price will mean less profit. Overall though we see there always seems to be a rather significant range over which the price can vary.

4 Conclusion

We have seen in Section 2 that under the assumption of complete markets there are examples of verification theorems under otherwise quite general assumptions. Given some decent methods for estimating the model parameters we can imagine this as being useful certain investors of the real market. However in Section 3 we argued that a real market will in general not be complete and the issue of completeness can complicate the analysis.

First we had the problem regarding the Hamilton-Jacobi-Bellman equation, which in the case of an incomplete market can no longer be reduced to a linear partial differential

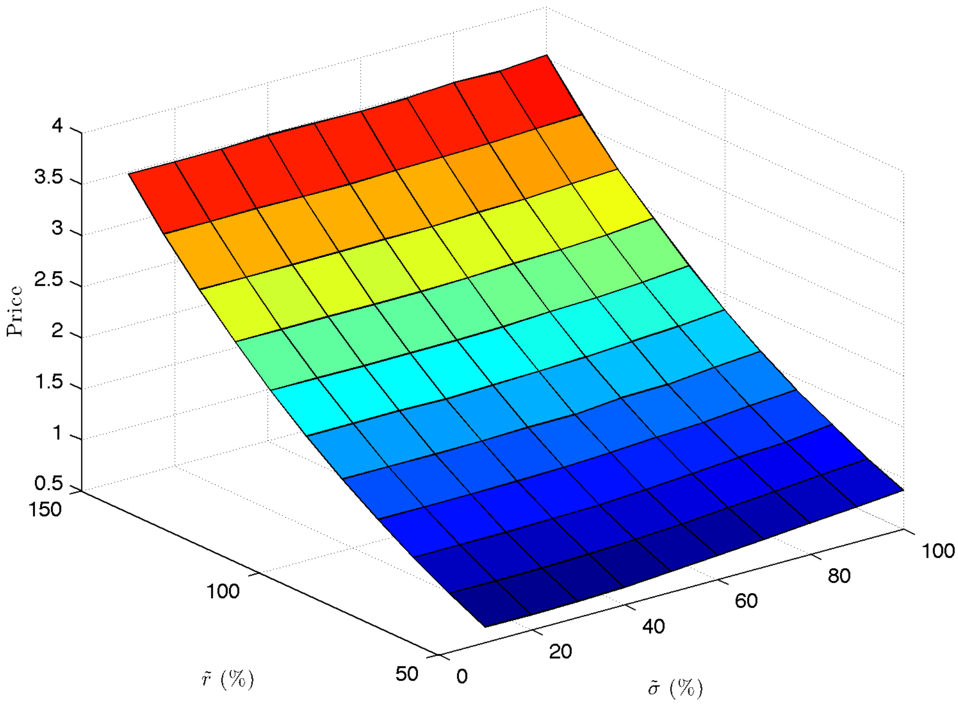

Figure 2: Option prices for $\tilde{S}(0)$ and $K = 1$.

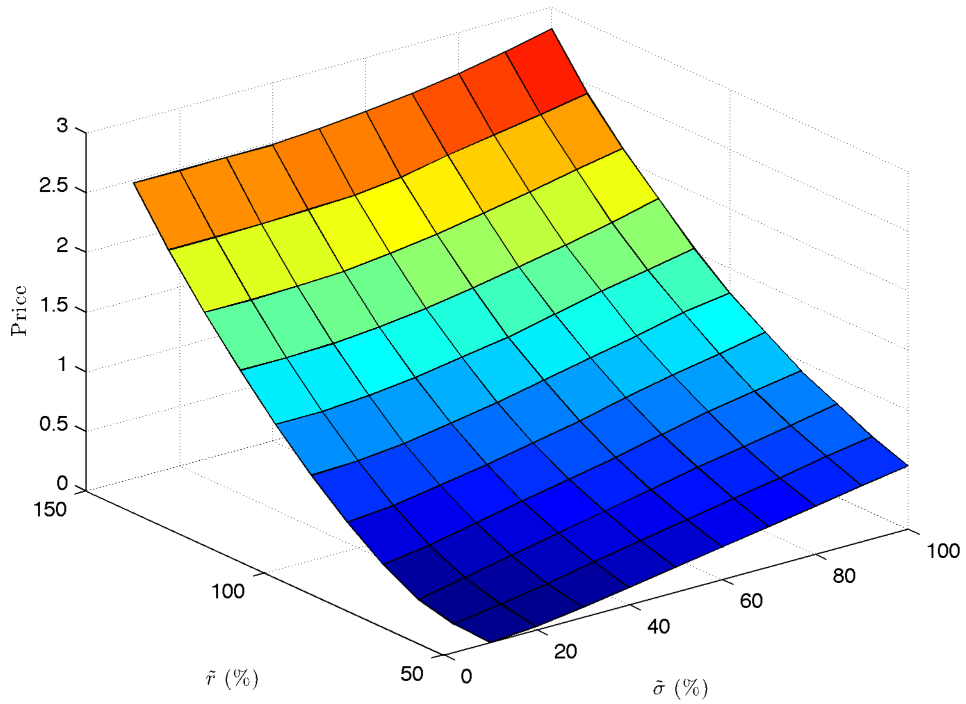

Figure 3: Option prices for $\tilde{S}(0)$ and $K = 2$.

equation. The proof of verification in Section 2 relied heavily on the existence of an explicit solution to the PDE to provide estimates on the portfolio. In the incomplete case the resulting nonlinear equation can no longer be solved analytically and we concluded that much more general methods must be used to examine the question of existence and optimality of solutions to the HJB equation.

Furthermore, we saw that the problem with incompleteness is more far-reaching than just absence of proper verification procedures. When trying to optimize a portfolio in a market we need to be able to make good predictions on how the assets will behave for future times. However if the market is incomplete there will always be an ambiguity regarding pricing. Most notably we saw this in the case of a call option on a non-tradable asset, where the option could take any value between 0 and ∞. Even if we were to prove verification, the ambiguity in pricing would still be a problem when trying to implement the optimization in practice.

5 Acknowledgments

I would like to express my gratitude to Professor Maciej Klimek, my supervisor, for arranging this project and for providing the necessary guidelines for finishing it. The time he has given has been very much appreciated.

References

[1] Bensoussan, A. (1992) *Stochastic control of partially observable systems*, Cambridge University Press, Cambridge.

[2] Björk, T. (2004) *Arbitrage theory in continuous time*, Oxford University Press, Oxford.

[3] Honda, T. & Kamimura, S. (2010) On the Verification Theorem of Dynamic Portfolio-Consumption Problems with Stochastic Market Price of Risk. *Asia-Pacific Finan Markets* (2011) 18:151-166.

[4] Hubalek, F. & Schachermayer, W. (2001) The limitations of no-arbitrage arguments for real options, *International Journal of Theoretical and Applied Finance 4(2)*, 361-373.

[5] Lipster, R. S., & Shiryaev A. N. (2001) *Statistics of random processes I: General theory, 2nd ed.* Springer Verlag, New York.

[6] Merton, R. (1969) Lifetime portfolio selection under uncertainty: The continuous-time case *Review of Economics and Statistics, 51*, 247-257.

[7] Merton, R. (1971) Optimum consumption and portfolio rules in a continuous-time model *Journal of Economic theory, 3*, 373-413.