Numerical simulation of the nonlinear Korteweg-de Vries equation using SBP in time

Diploma work (15 HP) in scientific computing

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Abstract

The first main focus in the present project is to analyse the boundary treatment of the nonlinear Korteweg-de Vries (KdV) equation. The second main focus is to derive a suitable numerical solution using a high-order finite difference method. Since the model involve third derivatives in space, the numerical treatment of boundaries is highly nontrivial. To aid the boundary treatment we will employ recently derived high-order accurate third derivative finite difference operators. The closures are based on the summation-by-parts (SBP) framework, thereby guaranteeing linear stability. The boundary conditions are imposed using a penalty technique, leading to a nonlinear ODE system. Since the non-linear KdV equation leads to very stiff ODE systems we will compare the traditional way of time-integration using the fourth order Runge-Kutta method with a novel SBP-SAT technique for time discretisation of the non-linear ODE system.

1 Motivation

A robust and well-proven high-order finite difference methodology that ensures the strict stability of time-dependent partial differential equations (PDEs) is the SBP-SAT method. The SBP-SAT method combines semi-discrete operators that satisfy a summation-by-parts (SBP) formula [2], with physical boundary conditions implemented using the simultaneous approximation term (SAT) method [1].

The SBP-SAT approach has so far been developed for problems involving first and second derivatives in space. However, there are many problems where higher-order derivatives are present. Some examples include the Korteweg-de Vries and the Boussinesq equations (describing nonlinear water waves), soliton models in neuroscience [5], the Euler-Lagrange equation for beams, and the Cahn-Hilliard equation which describes the process of phase separation.

The main focus in the present study is to construct a high-order accurate SBP-SAT approximation of the non-linear Korteweg-de Vries equation,

$$u_t = u_{xxx} + 6uu_x \ .$$

The following soliton solution

$$u(x,t) = \frac{1}{2}c \ sech^2\left(\frac{\sqrt{c}}{2}(x - ct - a)\right)$$

is an analytic solution to KdV. c is the wave speed and a an arbitrary constant.

$$sech(x) = \frac{2 \ e^{-x}}{1 + e^{-2 \ x}}$$

The main mathematical and numerical difficulty of this model comes from the boundary treatment due to the third derivative term. Another numerical difficulty comes from the fact that this model leads to extremely stiff semi-discrete models, after deriving a stable semi-discrete SBP-SAT model. For details concerning the spatial discretisation see [4]. The SBP-SAT treatment is traditionally only used to discretise the spatial derivatives, including the boundary conditions which leads no a system of non-linear ODE. The focus in the present project is to find an efficient time-integration for the nonlinear ODE system, by employing a novel SBP-SAT technique (see [3] for details) to discretise the time-derivative. To validate the efficiency we will compare against the traditional technique of employing the fourth order accurate Runge-Kutta method. The accuracy and stability properties will be investigated mathematically using the energy method and later verified against analytical solutions.

2 Project plan and time frame

- Start by a literature study to learn about the Korteweg-de Vries equation model (and other dispersive wave equation models) to understand the underlying physics and applications (1 weeks).
- Analyze well-posedness using the energy method for the continuous model (1 weeks).
- Derive a stable semi-discrete SBP-SAT approximation of well-posed problems above (1 weeks).

- Derive a stable SBP-SAT approximation (in time) of a linearised version of the the non-linear ODE system in the previous step (1 weeks).
- Derive a stable SBP-SAT approximation (in time) of the non-linear ODE system in the previous step (1 weeks). **Remark: This will be done if enough time!**
- Verify the stability and accuracy against analytic solution solutions (1 weeks).
- Complete the report (3 weeks).

$$v_{tt} = H(v, v_t, t)$$

References

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