



UPPSALA  
UNIVERSITET

# Numerical simulation of solitons

---

Ludvig Lindeberg and Tuan Anh Dao

**Supervisor:** Ken Mattsson

**Project in Computational Science: Report**

January 2019

PROJECT REPORT





# Numerical simulation of solitons

Ludvig Lindeberg and Tuan Anh Dao

January 2019

## Abstract

We analyse numerically the periodic problem and the initial boundary value problem of the Korteweg-de Vries equation and the Drinfeld-Sokolov-Wilson equation using the summation-by-parts simultaneous-approximation-term (SBP-SAT) method. Two sets of boundary conditions are derived for each equation of which stability is shown using the energy method. Numerical analysis is done when the solution interacts with the boundaries. Results show the benefit of higher order SBP operators.

**Keywords:** finite difference methods, boundary treatment, solitons

## 1 Introduction

*Solitons* (or *solitary waves*) are special kinds of waves with a distinct set of features. Solitons arise due to a balance between nonlinear effects and dispersive effects. Thus, soliton solutions can be found for nonlinear dispersive partial differential equations (PDEs). Solitons are characterized by the following: (i) when propagating at a constant speed, the shape of the soliton remains unchanged; (ii) when a collision occurs between two solitons, both solitons emerge unchanged. In this project, we study two different PDEs that both exhibit soliton solutions [7].

Originated in the nineteenth century but not until the sixties when the Korteweg-de Vries equation (KdV equation) became a fruitful research object for the scientists after they realized its apparent benefits in various applications [2]. The early-known soliton solution of the equation mimics an observed phenomenon of water waves induced by boat engine and initially called by its father the “Waves of Translation”. In modern application, the equation is most used to describe the translation of waves with weak dispersion such as the waves in shallow water which have small amplitude and long wavelength; or the movement of internal flows in some special fluid [9]. It is also helpful in the construction of various complex models such as plasma waves, shock waves, tornados, hydrodynamics, etc. Since an exact solution can be derived, the KdV equation is usually useful to verify the accuracy of numerical methods.

An extension of the KdV equation is the coupled Drinfel’d-Sokolov-Wilson equation, which we refer to as the DSW equation. The DSW equation is a coupled system of two nonlinear dispersive PDEs, independently proposed by Drinfel’d and Sokolov [4] and Wilson [13]. Just like the KdV equation, it can be used to model the translation shallow water waves. Interestingly, in addition to the regular soliton solutions found for the KdV equation, this system exhibits a new class of solutions referred to as “static solitons”, which behave similarly to regular solitons [6]. In this paper, however, we focus on the regular soliton solutions.

Boundary treatment has long been known to pose a major challenge to time-dependent PDEs. A robust solution approach is to approximate the governing equation using summation-by-parts (SBP) operators [8] and impose the boundary conditions with simultaneous approximation term (SAT) method [3], that we refer as the SBP-SAT method. A natural advantage of the SBP-SAT method is that it comes with well-controlled stability and high order of accuracy, which is known to provide more computational efficiency and able to capture more complex phenomena [1, 11, 12]. In a recent work on the nerve soliton [11], the method reveals its suitability for solving equations of the same type to this work. However, since our objective is a coupled system of nonlinear equations, the difficulty still consists in the boundary treatment since it is nontrivial to derive a set

of well-posed boundary conditions as well as a stable corresponding SAT scheme. Our main tool for the mathematical analysis is the *energy method* (see [5, 12]). It is used to verify well-posedness for the continuous problems and stability for the discrete problems. For time integration, the fourth order accurate Runge-Kutta method is used.

In Section 2, we study well-posedness for the KdV equation and derive stable SBP-SAT approximations for two sets of boundary conditions. Then we continue with the coupled DSW equation in Section 3, analysing well-posedness for the continuous equation and deriving stable SBP-SAT approximations for two sets of boundary conditions. Tables containing convergence analysis are presented as well as efficiency plots. Finally, the obtained results are discussed.

## 2 The Korteweg-de Vries equation

The KdV equation in its most known form can be written as

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1)$$

where the subscript notations denote the partial derivatives that are used for the rest of this report. The function  $u$  depending on the one-dimensional space variable  $x$  and the time variable  $t$  characterizes the elongation of the wave at point  $x$  and time  $t$  [2]. The second and third term in the left hand side of (1) is called the nonlinear term and the dispersion term, respectively.

An exact solution of (1) is given by (cf, [9])

$$u_{\text{exact}}(x, t) = \frac{c}{2} \operatorname{sech}^2 \left( \frac{\sqrt{c}(x - ct - a)}{2} \right), \quad (2)$$

where  $c$  is the wave speed and  $a$  is the starting offset. We use the above exact solution to construct the initial condition and calculate the errors for the efficiency analysis.

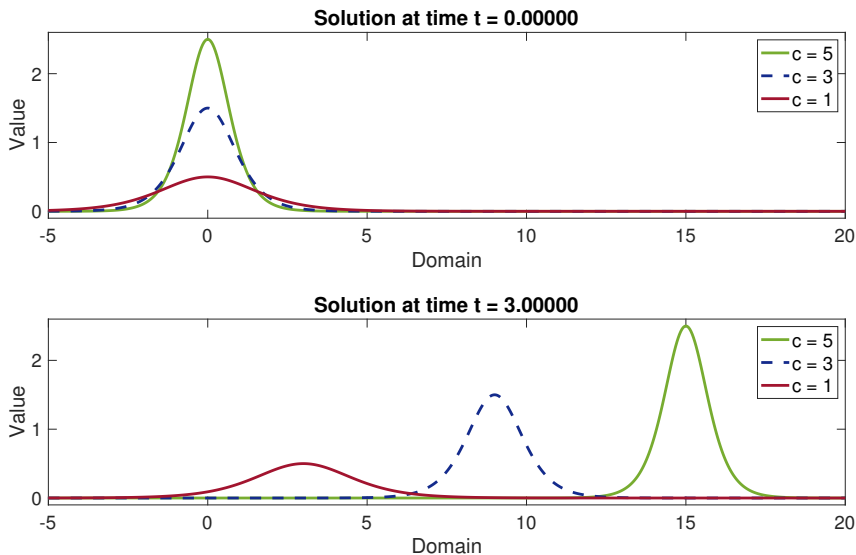


Figure 1: The soliton solution (2) of the KdV equation at two different time points with different choices of the wave speed

From (2), it is clear that to have a real solution, the wave speed  $c$  must be a non-negative number. Figure 1 illustrates the soliton solution given by (2) for several positive values of  $c$ . It can be observed that with those choices of  $c$ , the soliton propagates to the right; and its amplitude is proportional to  $c$ . In other words, the solitary waves with higher amplitude move faster than the ones with smaller amplitude.

## 2.1 Well-posedness

**Definition 2.1.** Given two continuous functions  $u$  and  $v$ , a continuous inner product is defined as  $(u, v) = \int_{x_l}^{x_r} u^T v dx$  and its corresponding norm is  $\|u\|^2 = (u, u) = \int_{x_l}^{x_r} u^T u dx$ .

To analyse the well-posedness, we use the so-called “energy method”. Given an equation in the form  $u_t = f(u, u_x, \dots)$ , we perform the method by multiplying both sides of the equation by  $u^T$  and integrating by parts to obtain  $(u, u_t)$ . We then add the transpose  $(u_t, u)$ . For the problem to be well-posed, it requires that there is no energy growth in time. Due to the relation  $\frac{d}{dt} \|u\|^2 = (u_t, u) + (u, u_t)$ , we can investigate this.

We begin by writing (1) on skew-symmetric form, yielding

$$u_t + 3\alpha(u^2)_x + 6(1 - \alpha)uu_x + u_{xxx} = 0, \quad (3)$$

for some constant  $\alpha$ . Now we apply the energy method to (3),

$$\begin{aligned} (u, u_t) &= -(u, u_{xxx}) - 3\alpha(u, (u^2)_x) - 6(1 - \alpha)(u, uu_x) - (u, u_{xxx}) \\ &= (u_{xxx}, u) + 3\alpha((u^2)_x, u) + 6(1 - \alpha)(u_x, u^2) + (u_x^2 - 2uu_{xx})|_{x_l}^{x_r} \\ &\quad - ((3\alpha + 6(1 - \alpha))u^3)|_{x_l}^{x_r}. \end{aligned}$$

Adding the transpose  $(u_t, u)$  gives

$$\begin{aligned} \frac{d}{dt} \|u\|^2 &= (u_x^2 - 2uu_{xx})|_{x_l}^{x_r} - ((3\alpha + 6(1 - \alpha))u^3)|_{x_l}^{x_r} + (6(1 - \alpha) - 3\alpha)(u_x, u^2) \\ &\quad + (3\alpha - 6(1 - \alpha))((u^2)_x, u). \end{aligned}$$

Setting  $\alpha = \frac{2}{3}$  gives us the nonlinear energy estimate

$$\frac{d}{dt} \|u\|^2 = (u_x^2 - 2uu_{xx} - 4u^3)|_{x_l}^{x_r}. \quad (4)$$

For periodic boundary conditions, we have  $\frac{d}{dt} \|u\|^2 = 0$ , and (3) is thus well-posed.

## 2.2 Stability

Some necessary discrete operators are restated in Definition 2.2 and 2.3 (see [10] for other details on this approximation technique).

**Definition 2.2.** A finite difference operator  $D_1 = H^{-1}(Q + \frac{1}{2}B)$ ,  $B = -e_1 e_1^T + e_m e_m^T$  approximating  $\frac{\partial}{\partial x}$  is said to be a first derivative SBP operator if the matrix  $H$  is symmetric positive definite and  $Q + Q^T = 0$ .

**Definition 2.3.** A finite difference operator  $D_3 = H^{-1}(R + \frac{1}{2}d_{1,1}^T d_{1,1} - \frac{1}{2}d_{1,m}^T d_{1,m} - e_1 d_{2,1} + e_m d_{2,m})$  approximating  $\frac{\partial^3}{\partial x^3}$  is said to be a third derivative SBP operator if the matrix  $H$  is symmetric positive definite,  $R + R^T = 0$ ,  $d_{1,1}, d_{1,m}$  approximate the first derivatives at the left and right boundary points, and  $d_{2,1}, d_{2,m}$  approximate the second derivatives at the left and right boundary points, respectively.

**Definition 2.4.** A discrete diagonal norm corresponding to the continuous norm given in Definition 2.1 is defined as

$$\|v\|_H^2 = v^T H v.$$

Analogously to the continuous case, the energy method can also be applied to an approximation in the form  $v_t = F(v)$ , where  $v$  is a discrete variable. In this case, we multiply both sides of the equation by  $v^T H$  to obtain  $v^T H v_t$  and add the transpose  $v_t^T H v$ . For the approximation to be stable, it requires that there is no energy growth in time. Due to the relation  $\frac{d}{dt} \|v\|^2 = v^T H v_t + v_t^T H v$ , we can investigate this.

A consistent finite difference approximation of (1) with periodic boundary condition (ignoring the boundary terms) can be written as

$$v_t + 2D_1\bar{v}v + 2\bar{v}D_1v + D_3v = 0, \quad (5)$$

where the discrete function  $v = [v_1, v_2, \dots, v_n]$  approximates  $u$  on  $n$  grid points; the diagonal matrix

$\bar{v} = \begin{bmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_m \end{bmatrix}$ . We now prove that the above approximation is stable. Multiplying both sides of (5) by  $v^T H$  gives

$$v^T H v_t = -2v^T H D_1 \bar{v} v - 2v^T H \bar{v} D_1 v - v^T H D_3 v.$$

Assuming that  $H$  is diagonal, we have  $H\bar{v} = \bar{v}H$ . It deduces

$$\begin{aligned} v^T H v_t &= -2v^T \left( Q + \frac{1}{2}B \right) \bar{v} v - 2v^T \bar{v} \left( Q + \frac{1}{2}B \right) v \\ &\quad - v^T \left( R + \frac{1}{2}d_{1;1}^T d_{1;1} - \frac{1}{2}d_{1;m}^T d_{1;m} - e_1 d_{2;1} + e_m d_{2;m} \right) v. \end{aligned}$$

We then add the transpose

$$\begin{aligned} v_t^T H v &= -2(\bar{v}v)^T \left( Q^T + \frac{1}{2}B \right) v - 2v^T \left( Q^T + \frac{1}{2}B \right) \bar{v} v \\ &\quad - v^T \left( R^T + \frac{1}{2}d_{1;1}^T d_{1;1} - \frac{1}{2}d_{1;m}^T d_{1;m} - e_1 d_{2;1} + e_m d_{2;m} \right) v \end{aligned}$$

to obtain

$$\begin{aligned} \frac{d}{dt} \|v\|_H^2 &= -2v^T (e_m e_m^T - e_1 e_1^T) \bar{v} v - 2(\bar{v}v)^T (e_m e_m^T - e_1 e_1^T) v \\ &\quad - v^T (d_{1;1}^T d_{1;1} - d_{1;m}^T d_{1;m} - 2e_1 d_{2;1} + 2e_m d_{2;m}) v \\ &= -4v_m^3 + 4v_1^3 + (d_{1;m}v)^2 - (d_{1;1}v)^2 \\ &\quad - 2v_m d_{2;m} v + 2v_1 d_{2;1} v. \end{aligned}$$

It is obvious that the above approximation is stable, because  $\frac{d}{dt} \|v\|_H^2 = 0$  for periodic boundary conditions (i.e.,  $v_1 = v_m; d_{1;1}v = d_{1;m}v; d_{2;1}v = d_{2;m}v$ ).

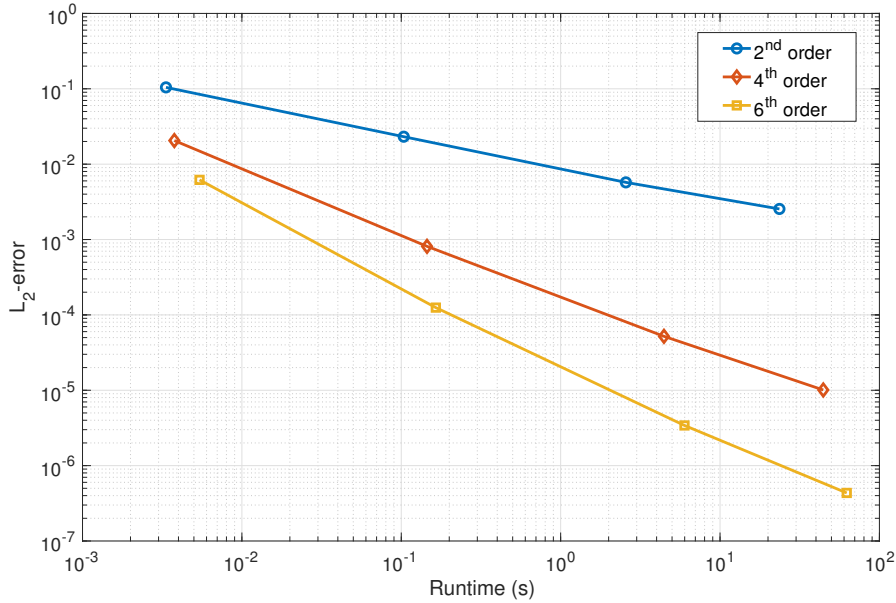


Figure 2:  $L_2$ -error as a function of the runtime

An analysis of runtime-error efficiency is performed on the domain  $[x_l, x_r] = [-20, 20]$  with  $m = 51, 101, 201, 301$  grid points, wave speed  $c = 2$ , time step size  $\Delta t = \beta h^3$  where  $h = (x_r - x_l)/m$ . Notice that the severe time step scaling  $\mathcal{O}(h^3)$  comes from the third derivative in (1). The result is shown in Figure 2. For a fair comparison among different orders of accuracy, sharp estimates of the condition number  $\beta$  as shown in Table 1 are used. For each accuracy order, starting from a sufficiently large value, the condition number  $\beta$  is computed by manually decreasing it until the solution becomes unstable.

	2 <sup>nd</sup> order	4 <sup>th</sup> order	6 <sup>th</sup> order
$\beta$	1.09	0.61	0.45

Table 1: Sharp estimates of  $\beta$  for second-, fourth- and sixth-order operators with  $m = 201$

### 2.2.1 A first set of well-posed boundary conditions

The following set of boundary conditions yield a well-posed problem for (1)

$$\begin{cases} u_l = g_l(t) \\ u_r = g_r(t) \\ (u_x)_r = \tilde{g}_r(t) \end{cases} \quad (6)$$

Applying these BCs to (4) leads to,

$$\begin{aligned} \frac{d}{dt} \|u\|^2 &= \tilde{g}_r^2(t) - 2g_r(u_{xx} + 2g_r^2) - u_x^2 + 2g_l(u_{xx} + 2g_l^2) \\ &= \tilde{g}_r^2(t) - 2g_r u_{xx} - 4g_r^3 - u_x^2 + 2g_l u_{xx} + 4g_l^3. \end{aligned}$$

We see that for zero boundary data,  $g_l(t) = g_r(t) = \tilde{g}_r = 0$ , (1) is a well-posed problem. However, since the energy is not bounded by the boundary data, (6) are not strongly well-posed BCs (see [5] for the definition of strong well-posedness).

A stable SBP-SAT approximation of (1) subjected to (6) with zero boundary data is,

$$\begin{aligned}
v_t &= -2D_1\bar{v}v - 2\bar{v}D_1v - D_3v \\
&+ \tau_1 H^{-1}e_1v_1(v_1 - 0) + \tau_2 H^{-1}e_mv_m(v_m - 0) \\
&+ \tau_3 H^{-1}d_{2;m}^T(v_m - 0) + \tau_4 H^{-1}d_{1;m}^T(d_{1;m}v - 0) + \tau_5 H^{-1}d_{2;1}^T(v_1 - 0).
\end{aligned} \tag{7}$$

The energy method applied to (7) gives the following energy estimate:

$$\begin{aligned}
\frac{d}{dt} \|v\|_H^2 &= -4v_m^3 + 4v_1^3 + (d_{1;m}v)^2 - (d_{1;1}v)^2 - 2v_md_{2;m}v + 2v_1d_{2;1}v \\
&+ 2\tau_1v_1^3 + 2\tau_2v_m^3 + 2\tau_3v_md_{2;m}v + 2\tau_4(d_{1;m})^2 + 2\tau_5v_1d_{2;1}v \\
&= (4 + 2\tau_1)v_1^3 + (2\tau_2 - 4)v_m^3 + (2\tau_3 - 2)v_md_{2;m}v \\
&+ (2\tau_4 + 1)(d_{1;m})^2 + (2\tau_5 + 2)v_1d_{2;1}v - (d_{1;1})^2.
\end{aligned}$$

We choose the penalty terms  $\tau_1 = -2, \tau_2 = 2, \tau_3 = 1, \tau_4 \leq -\frac{1}{2}, \tau_5 = -1$  to obtain the final energy estimate

$$\frac{d}{dt} \|v\|_H^2 = -(d_{1;1})^2 + (2\tau_4 + 1)(d_{1;m})^2, \tag{8}$$

which mimics the continuous case for homogeneous boundary conditions.

### 2.2.2 A set of strongly well-posed boundary conditions

We show that the following set of boundary conditions yields a strongly well-posed problem subjected to (1).

$$\begin{cases} 2(u_l + |u_l|)u_l + (u_{xx})_l = g_l(t), \\ 2(u_r - |u_r|)u_r + (u_{xx})_r = g_r(t), \\ (u_x)_r = \tilde{g}_r(t). \end{cases} \tag{9}$$

Denote  $u^\pm = \frac{u \pm |u|}{2}$ , it naturally deduces  $u = u^+ + u^-, u + |u| = 2u^+, u - |u| = 2u^-$  and  $u^+ - u^- = |u|$ . (9) can then be rewritten as

$$\begin{cases} 4u_l^+ u_l + (u_{xx})_l = g_l(t), \\ 4u_r^- u_r + (u_{xx})_r = g_r(t), \\ (u_x)_r = \tilde{g}_r(t). \end{cases}$$

Recall that

$$\begin{aligned}
\frac{d}{dt} \|u\|^2 &= (u_x)_r^2 - (u_x)_l^2 - 2u_r(2u_r^2 + (u_{xx})_r) + 2u_l(2u_l^2 + (u_{xx})_l) \\
&= (u_x)_r^2 - (u_x)_l^2 - 2u_r(2u_r(u_r^+ + u_r^-) + (u_{xx})_r) \\
&\quad + 2u_l(2u_l(u_l^+ + u_l^-)) + (u_{xx})_l.
\end{aligned}$$

Inserting (9) gives

$$\begin{aligned}
\frac{d}{dt} \|u\|^2 &= \tilde{g}_r^2(t) - (u_x)_l^2 - 2u_r(2u_ru_r^+ - 2u_ru_r^- + g_r) + 2u_l(-2u_lu_l^+ + 2u_lu_l^- + g_l) \\
&= \tilde{g}_r^2(t) - (u_x)_l^2 - 2u_r(2u_r|u_r| + g_r) + 2u_l(-2u_l|u_l| + g_l) \\
&= \tilde{g}_r^2(t) - (u_x)_l^2 - |u_r| \left( 4u_r^2 + 2u_r \frac{g_r}{|u_r|} \right) - |u_l| \left( 4u_l^2 - 2u_l \frac{g_l}{|u_l|} \right) \\
&= \tilde{g}_r^2(t) - (u_x)_l^2 - |u_r| \left( 2u_r + \frac{g_r}{2|u_r|} \right)^2 + \frac{g_r^2}{4|u_r|} \\
&\quad - |u_l| \left( 2u_l - \frac{g_l}{2|u_l|} \right)^2 + \frac{g_l^2}{4|u_l|}.
\end{aligned}$$



It is obvious that all the terms in the right hand side are either bounded or less than or equal to zero when  $|u_l| \gg 0, |u_r| \gg 0$ . However, even if  $|u_l| \rightarrow 0$  we still have  $\lim_{|u_l| \rightarrow 0} |u_l| \left(2u_l - \frac{g_l}{2|u_l|}\right)^2 + \frac{g_l^2}{4|u_l|} = 0$ . Similar argument can be applied for  $u_r$  to deduce that the right hand side is still bounded. Therefore, it is sufficient to conclude that (9) is a set of strongly well-posed boundary conditions for (1).

We now find the value of the real numbers  $\tau_l, \tau_r, \sigma_r$  such that the following SBP-SAT approximation of (1) subjected to (9) is stable,

$$\begin{aligned} v_t + 2D_1 \bar{v}v + 2\bar{v}D_1 v + D_3 v &= \tau_l H^{-1} e_1 (2(v_1 + |v_1|)v_1 + d_{2;1}v - g_l) \\ &+ \tau_r H^{-1} e_m (2(v_m - |v_m|)v_m + d_{2;m}v - g_r) \\ &+ \sigma_r H^{-1} d_{1;m}^T (d_{1;m}v - \tilde{g}_r). \end{aligned} \quad (10)$$

**Lemma 2.5.** *Eq. (10) is strongly stable if*

$$\tau_l = -1; \tau_r = 1; \sigma_r \leq -\frac{1}{2}.$$

*Proof.* The energy method applied to (10) leads to,

$$\begin{aligned} \frac{d}{dt} \|v\|_H^2 &= -4v_m^3 + 4v_1^3 + (d_{1;m}v)^2 - (d_{1;1}v)^2 - 2v_m d_{2;m}v + 2v_1 d_{2;1}v \\ &+ 2\tau_l v_1 (4v_1^+ v_1 + d_{2;1}v - g_l) + 2\tau_r v_m (4v_m^- v_m + d_{2;m}v - g_r) \\ &+ 2\sigma_r d_{1;m}v (d_{1;m}v - \tilde{g}_r) \\ &= -4v_m^2 (v_m^+ + v_m^- - 2\tau_r v_m^-) + 2\tau_l v_1 (d_{2;1}v - g_l) \\ &+ 4v_1^2 (v_1^+ + v_1^- + 2\tau_l v_1^+) + 2\tau_r v_m (d_{2;m}v - g_r) \\ &- (d_{1;1}v)^2 + (2\sigma_r + 1)(d_{1;m}v)^2 - 2\sigma_r d_{1;m}v \tilde{g}_r \\ &- 2v_m d_{2;m}v + 2v_1 d_{2;1}v. \end{aligned}$$

To cancel the cubic terms and the terms involving second derivatives, we choose  $\tau_l = -1, \tau_r = 1$ . The equation becomes

$$\begin{aligned} \frac{d}{dt} \|v\|_H^2 &= -4v_m^2 |v_m| - 4v_1^2 |v_1| + 2v_1 g_l - 2v_m g_r \\ &- (d_{1;1}v)^2 + (2\sigma_r + 1)(d_{1;m}v)^2 - 2\sigma_r \tilde{g}_r d_{1;m}v. \end{aligned} \quad (11)$$

Since the first part of the right hand side of (11) reads as

$$\begin{aligned} -4v_m^2 |v_m| - 4v_1^2 |v_1| + 2v_1 g_l - 2v_m g_r &= -|v_m| \left(2v_m + \frac{g_r}{2|v_m|}\right)^2 + \frac{g_r^2}{4|v_m|} \\ &- |v_1| \left(2v_1 - \frac{g_l}{2|v_1|}\right)^2 + \frac{g_l^2}{4|v_1|}, \end{aligned}$$

we apply the same argument as for the analysis of (9) to show that it is bounded. It is easy to see that for  $\sigma_r \leq -\frac{1}{2}$ , the remaining part of (11) is also bounded. For example, when  $\sigma_r = -1$ ,

$$\begin{aligned} -(d_{1;1}v)^2 + (2\sigma_r + 1)(d_{1;m}v)^2 - 2\sigma_r \tilde{g}_r d_{1;m}v &= -(d_{1;1}v)^2 - (d_{1;m}v)^2 - 2\tilde{g}_r d_{1;m}v \\ &= -(d_{1;1}v)^2 - (d_{1;m}v + \tilde{g}_r)^2 + \tilde{g}_r^2 \end{aligned}$$

contains all bounded or non-positive terms. □

### 3 The coupled Drinfeld-Sokolov-Wilson Equation

Now we consider a system of equations, namely the coupled Drinfeld-Sokolov-Wilson equation, or the DSW equation,

$$\begin{aligned} u_t + 3vv_x &= 0 \\ v_t + vu_x + 2uv_x + 2v_{xxx} &= 0, \end{aligned} \quad (12)$$

which is a coupled system of PDEs. The system consists of several nonlinear terms and a single dispersive, third order term.

An exact solution of (12) is given in [14] and is as follows,

$$\begin{aligned} u(x, t) &= \frac{3c}{2} \operatorname{sech}^2 \left( \sqrt{\frac{c}{2}}(x - ct - a) \right) \\ v(x, t) &= c \operatorname{sech} \left( \sqrt{\frac{c}{2}}(x - ct - a) \right), \end{aligned}$$

where  $c$  is the wave speed and  $a$  is the starting offset. An illustration is given in Figure 3 showing the soliton solution at two different time points. We use the above exact solution to construct the initial condition and calculate the errors for the efficiency and convergence analysis.

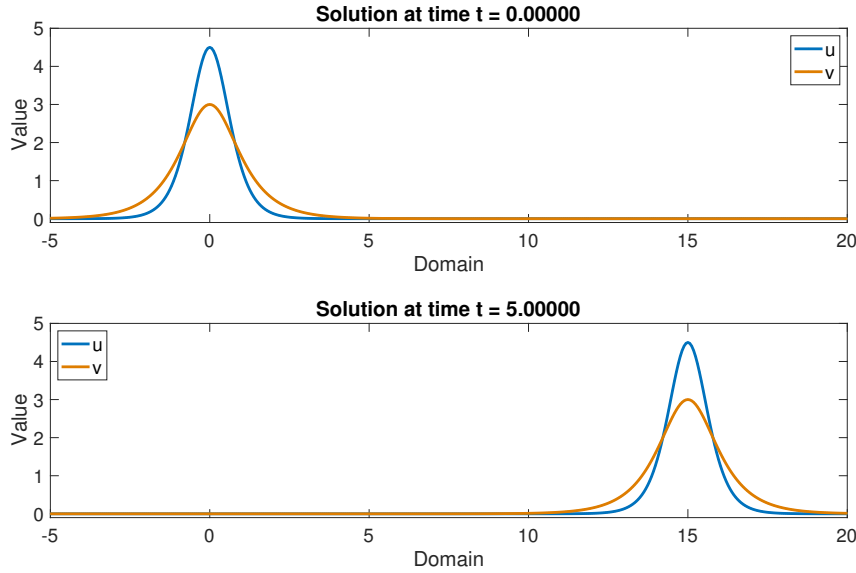


Figure 3: Soliton solution at time  $t = 0$  and  $t = 3$  with the choice of parameters  $c = 2, a = 0$

We begin by applying the energy method to (12) in order to analyse well-posedness. Multiplying the first equation by  $u$  and the second equation by  $v$  and IBP leads to,

$$\begin{aligned} (u, u_t) &= -3(uv, v_x) = -3uv^2|_{x_l}^{x_r} + 3((uv)_x, v) \\ (v, v_t) &= -(v^2, u_x) - 2(uv, v_x) - 2(v, v_{xxx}) \\ &= \{-4vv_x + 2v_x^2 - 3uv^2\}|_{x_l}^{x_r} + 2(v_{xxx}, v) + ((v^2)_x, u) + 2((uv)_x, v). \end{aligned}$$

Adding the transpose gives

$$\begin{aligned} \frac{d}{dt} \|u\|^2 &= -3uv^2|_{x_l}^{x_r} + 3((uv)_x, v) - 3(v_x, uv) \\ \frac{d}{dt} \|v\|^2 &= \{-4vv_x + 2v_x^2 - 3uv^2\}|_{x_l}^{x_r} + ((v^2)_x, u) + 2((uv)_x, v) - (u_x, v^2) - 2(v_x, uv). \end{aligned}$$

Our final energy estimate of (12) is thus

$$\frac{d}{dt}(\|u\|^2 + \|v\|^2) = BT + RT, \quad (13)$$

where

$$BT = \{-4vv_{xx} + 2v_x^2 - 6uv^2\} \Big|_{x_l}^{x_r}$$

are the boundary terms and

$$RT = 5((uv)_x, v) - 5(v_x, uv) + ((v^2)_x, u) - (u_x, v^2)$$

are the remainder terms.  $BT$  can be rewritten as

$$BT = \underbrace{\begin{bmatrix} u \\ v \\ v_x \\ v_{xx} \end{bmatrix}}_{w^T} \underbrace{\begin{bmatrix} 0 & -3v & 0 & 0 \\ -3v & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}}_M \underbrace{\begin{bmatrix} u \\ v \\ v_x \\ v_{xx} \end{bmatrix}}_w \Big|_{x_l}^{x_r}. \quad (14)$$

For periodic boundary conditions, it is clear that the boundary terms cancel.

**Remark 1.** In (13), it is not possible to eliminate  $RT$  and we do not achieve a nonlinear energy estimate. Assuming smooth solutions (such as solitons) we expect that the Cauchy problem is well-posed since the corresponding linearized problem can be shown to be well-posed.

We define the linearized problem as

$$\begin{aligned} u_t + 3\tilde{v}v_x &= 0 \\ v_t + \tilde{v}u_x + 2\tilde{u}v_x + 2v_{xxx} &= 0, \end{aligned} \quad (15)$$

where  $\tilde{u}$  and  $\tilde{v}$  are constants. Applying the energy method leads to,

$$\begin{aligned} (u, u_t) &= -3\tilde{v}uv \Big|_{x_l}^{x_r} + 3\tilde{v}(u_x, v) \\ (u_t, u) &= -3\tilde{v}(v_x, u) \\ (v, v_t) &= (-\tilde{v}uv - 2\tilde{u}v^2 - 4vv_{xx} + 2v_x^2) \Big|_{x_l}^{x_r} + \tilde{v}(v_x, u) + 2\tilde{u}(v_x, v) + 2(v_{xxx}, v) \\ (v_t, v) &= -\tilde{v}(u_x, v) - 2\tilde{u}(v_x, v) - 2(v_{xxx}, v). \end{aligned}$$

We thus can obtain the energy for each component.

$$\begin{aligned} \frac{d}{dt} \|u\|^2 &= -3\tilde{v}uv \Big|_{x_l}^{x_r} + 3\tilde{v}(u_x, v) - 3\tilde{v}(v_x, u) \\ \frac{d}{dt} \|v\|^2 &= (-\tilde{v}uv - 2\tilde{u}v^2 - 4vv_{xx} + 2v_x^2) \Big|_{x_l}^{x_r} + \tilde{v}(v_x, u) - \tilde{v}(u_x, v) \end{aligned}$$

It can be seen that in the standard norm, we cannot obtain a strict energy estimate since the remaining terms (besides the boundary terms) cannot be controlled. However, in some other norm, it is possible to cancel all the remaining terms to obtain an estimate which only contains the boundary terms in a symmetric quadratic form

$$\frac{d}{dt} \left( \frac{1}{3} \|u\|^2 + \|v\|^2 \right) = \begin{bmatrix} u \\ v \\ v_x \\ v_{xx} \end{bmatrix}^T \begin{bmatrix} 0 & -\tilde{v} & 0 & 0 \\ -\tilde{v} & -2\tilde{u} & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ v_x \\ v_{xx} \end{bmatrix} \Big|_{x_l}^{x_r}. \quad (16)$$

A convergence analysis is performed on the domain  $[x_l, x_r] = [-30, 30]$  with  $m$  space grid points, the wave speed  $c = 2$ , time step size  $\Delta t = 0.2h^3$  where  $h = (x_r - x_l)/m$ . The result is shown in Table 2.

$m$	error <sup>(2nd)</sup>	$q^{(2nd)}$	error <sup>(4th)</sup>	$q^{(4th)}$	error <sup>(6th)</sup>	$q^{(6th)}$
101	0.2394	-	0.0579	-	0.0383	-
151	0.1113	1.9039	0.0115	4.0243	0.0021	7.2001
201	0.0633	1.9756	0.0036	4.0535	0.0004	5.7942

Table 2:  $L_2$ -norm of the error and convergence rate for the DSW problem with periodic boundary conditions by different accuracy orders of SBP operators and number of grid points. Solutions are examined at  $t = 5$ .

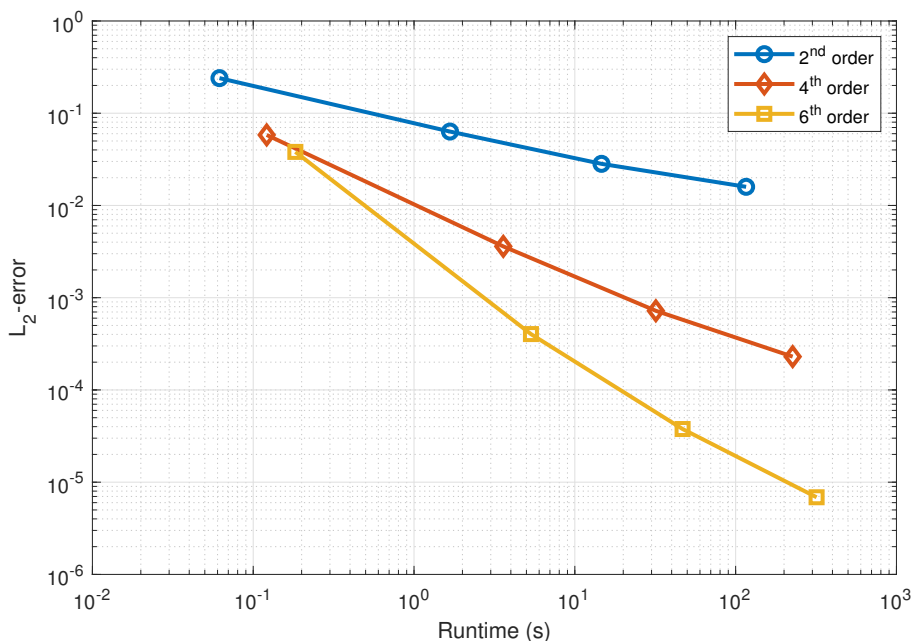


Figure 4:  $L_2$ -error as a function of the runtime.

An analysis of runtime-error efficiency is performed on the same domain and problem settings with different number of grid points. The result is shown in Figure 4. For a fair comparison among different orders of accuracy, sharp estimates of the condition number  $\beta$ , as shown in Table 3, are used. For each accuracy order, starting from a sufficiently large value, the condition number  $\beta$  is computed by manually decreasing it until the solution becomes unstable.

	2 <sup>nd</sup> order	4 <sup>th</sup> order	6 <sup>th</sup> order
$\beta$	0.54	0.30	0.22

Table 3: Sharp estimates of the CFL condition for  $m = 201$ .

### 3.1 A first set of well-posed boundary conditions

Considering zero boundary data (i.e., assuming  $v(x_l) = v(x_r) = 0$ ), it is clear that the boundary matrix  $M$  in (14) has three nonzero eigenvalues. The correct number of boundary conditions is

then three. Such a set of boundary conditions, for example, is

$$\begin{cases} v_l = g_l, \\ v_r = g_r, \\ (v_x)_r = \tilde{g}_r. \end{cases} \quad (17)$$

It is easy to see that inserting (17) with zero boundary data into both the energy estimates (14) and (16) makes the boundary terms less than or equal to zero. Therefore, (17) is a set of linearly well-posed boundary conditions. A consistent SBP-SAT approximation of (12) subjected to (17) is then

$$\begin{aligned} u_t &= -3\bar{v}D_1v + \tau_1H^{-1}e_1v_1(v_1 - g_l) + \tau_2H^{-1}e_mv_m(v_m - g_r) \\ v_t &= -\bar{v}D_1u - 2\bar{u}D_1v - 2D_3v + \sigma_1H^{-1}d_{2;m}^T(v_m - g_r) \\ &\quad + \sigma_2H^{-1}d_{1;m}^T(d_{1;m}v - \tilde{g}_r) + \sigma_3H^{-1}d_{2;1}^T(v_1 - g_l), \end{aligned} \quad (18)$$

where  $u, v$  are the discrete variables (here we use the same notation as the corresponding continuous ones) and  $\bar{u}, \bar{v}$  denote the square diagonal matrices of which the diagonals are  $u$  and  $v$ .

**Lemma 3.1.** *Eq. (18) is linearly stable if*

$$\tau_1 = -3; \tau_2 = 3; \sigma_1 = 2; \sigma_2 \leq -1; \sigma_3 = -2.$$

*Proof.* Applying the energy method to (18), we have

$$\begin{aligned} u^T H u_t &= -3u^T \bar{v} \left( Q + \frac{1}{2} B \right) v + \tau_1 u_1 v_1^2 + \tau_2 u_m v_m^2; \\ u_t^T H u &= -3v^T \left( Q^T + \frac{1}{2} B \right) \bar{v} u + \tau_1 u_1 v_1^2 + \tau_2 u_m v_m^2; \\ v^T H v_t &= -v^T \bar{v} \left( Q + \frac{1}{2} B \right) u - 2v^T \bar{u} \left( Q + \frac{1}{2} B \right) v - 2v^T H D_3 v \\ &\quad + \sigma_1 v_m d_{2;m} v + \sigma_2 (d_m v)^2 + \sigma_3 v_1 d_{2;1} v; \\ v_t^T H v &= -u^T \left( Q^T + \frac{1}{2} B \right) \bar{v} v - 2v^T \left( Q^T + \frac{1}{2} B \right) \bar{u} v - 2(D_3 v)^T H v \\ &\quad + \sigma_1 v_m d_{2;m} v + \sigma_2 (d_m v)^2 + \sigma_3 v_1 d_{2;1} v. \end{aligned}$$

Therefore, we have the discrete energy estimate

$$\begin{aligned} \frac{d}{dt} (\|u\|_H^2 + \|v\|_H^2) &= -5u^T \bar{v} \left( Q + \frac{1}{2} B \right) v - 5v^T \left( Q + \frac{1}{2} B \right) \bar{v} u \\ &\quad - v^T \bar{v} \left( Q + \frac{1}{2} B \right) u - u^T \left( Q + \frac{1}{2} B \right) \bar{v} v \\ &\quad + 2\tau_1 u_1 v_1^2 + 2\tau_2 u_m v_m^2 + 2\sigma_1 v_m d_{2;m} v + 2\sigma_2 (d_m v)^2 \\ &\quad + 2\sigma_3 v_1 d_{2;1} v - 2v^T H D_3 v - 2(D_3 v)^T H v, \end{aligned}$$

where the sum of the third derivatives can be simplified as

$$v^T H D_3 v + (D_3 v)^T H v = (d_{1;1} v)^2 - (d_{1;m} v)^2 - 2v_1 d_{2;1} v + 2v_m d_{2;m} v.$$

The boundary terms of the discrete energy estimate is then

$$\begin{aligned} \text{BT} &= -6u_m v_m^2 + 6u_1 v_1^2 \\ &\quad + 2\tau_1 u_1 v_1^2 + 2\tau_2 u_m v_m^2 + 2\sigma_1 v_m d_{2;m} v + 2\sigma_2 (d_{1;m} v)^2 + 2\sigma_3 v_1 d_{2;1} v \\ &\quad - 2(d_{1;1} v)^2 + 2(d_{1;m} v)^2 + 4v_1 d_{2;1} v - 4v_m d_{2;m} v \end{aligned}$$

which exactly mimics the continuous estimate. We choose the penalty parameters  $\tau_1 = -3, \tau_2 = 3, \sigma_1 = 2, \sigma_2 \leq -1, \sigma_3 = -2$  to obtain the final boundary term

$$\text{BT} = -2(d_{1;1} v)^2 + 2(1 + \sigma_2)(d_{1;m} v)^2 \leq 0.$$

Therefore, with the above choice of parameters, the approximation (18) is linearly stable.  $\square$

A convergence analysis is performed on the domain  $[x_l, x_r] = [-30, 30]$  with  $m$  space grid points, the wave speed  $c = 2$ , time step size  $\Delta t = 0.01h^3$  where  $h = (x_r - x_l)/(m - 1)$ . The free parameter  $\sigma_2$  is set to be  $-2$ . The result is shown in Table 4.

$m$	error <sup>(2nd)</sup>	$q$ <sup>(2nd)</sup>	error <sup>(4th)</sup>	$q$ <sup>(4th)</sup>	error <sup>(6th)</sup>	$q$ <sup>(6th)</sup>
101	0.2419	-	0.0558	-	0.0203	-
151	0.1125	1.9038	0.0114	3.9503	0.0022	5.5368
201	0.0637	1.9887	0.0037	3.9718	0.0004	5.8058

Table 4:  $L_2$ -norm of the error and convergence rate for the DSW problem with the set of boundary conditions (17) by different accuracy orders of SBP operators and number of space grid points. Solutions are examined at  $t = 5$ .

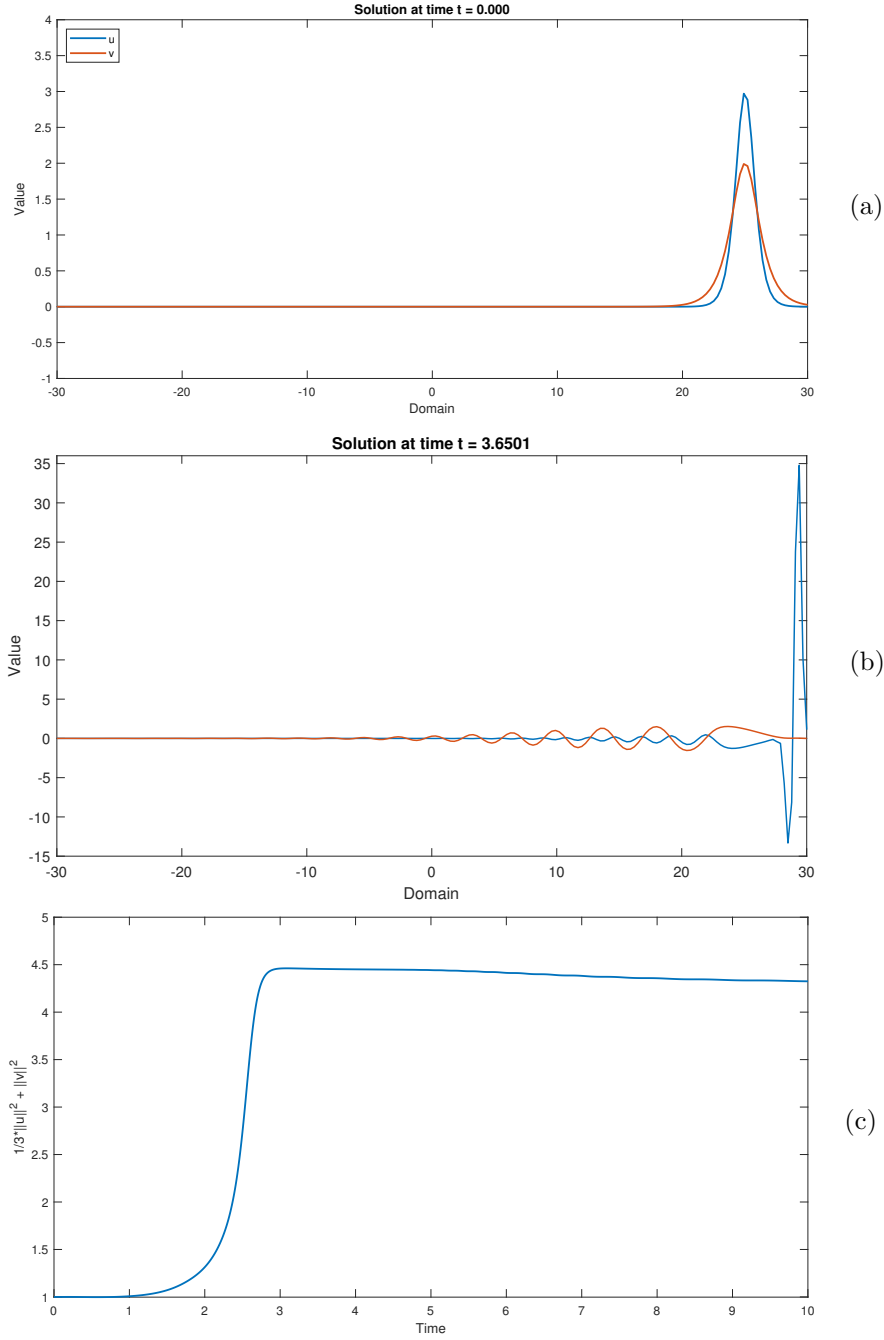


Figure 5: Boundary interaction at time  $t = 0$  and  $t = 10$ ,  $m = 201$ .

Two snapshots at different time points of the boundary interaction and the total energy over the course of interaction are shown in Figure 5. Because of the non-physics strong forcing  $v_l = v_r = (v_x)_r = 0$ , a strange behaviour of the solution after the interaction can be observed near the boundaries. Though we can see a growth in energy after the interaction, the total energy is still bounded as seen in the energy plot (Figure 5(c)).

### 3.2 Characteristic boundary conditions

Recall that  $\text{BT} = w^T M w$ , where  $w = (u, v, v_x, v_{xx})^T$ ,  $M = \begin{bmatrix} 0 & -3v & 0 & 0 \\ -3v & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$ . Since  $M$  is

symmetric, it can be diagonalized as

$$M = S M_s S^T,$$

where

$$M_s = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -\sqrt{9v^2 + 4} & 0 \\ 0 & 0 & 0 & \sqrt{9v^2 + 4} \end{bmatrix};$$

and

$$S = \begin{bmatrix} \frac{-2}{3v\sqrt{\frac{4}{9v^2} + 1}} & 0 & \frac{3v}{2\sqrt{\frac{9v^2}{2} + 2}} & \frac{3v}{2\sqrt{\frac{9v^2}{2} + 2}} \\ 0 & 0 & \frac{\sqrt{9v^2 + 4}}{2\sqrt{\frac{9v^2}{2} + 2}} & \frac{-\sqrt{9v^2 + 4}}{2\sqrt{\frac{9v^2}{2} + 2}} \\ 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{\frac{4}{9v^2} + 1}} & 0 & \frac{1}{\sqrt{\frac{9v^2}{2} + 2}} & \frac{1}{\sqrt{\frac{9v^2}{2} + 2}} \end{bmatrix} (S^T \equiv S^{-1}).$$

Let  $M_s^+ = \frac{M_s + |M_s|}{2}$ ,  $M_s^- = \frac{M_s - |M_s|}{2}$ ,  $M^+ = S M_s^+ S^T$ ,  $M^- = S M_s^- S^T$ . It is clear that  $M_s = M_s^+ + M_s^-$ ,  $M = M^+ + M^-$ , and  $M^+$ ,  $M^-$  are symmetric matrices  $(M^+)^T = (S M_s^+ S^T)^T = S M_s^+ S^T = M^+$ . A set of characteristic boundary conditions is

$$\begin{cases} M^+ w = M^+ g_r, \text{ at } x = x_r \\ M^- w = M^- g_l, \text{ at } x = x_l \end{cases} \quad (19)$$

where  $g_r, g_l$  are some  $4 \times 1$  vectors of time-dependent real functions.

We apply (19) to the boundary term given by (14) to obtain

$$\begin{aligned} \text{BT} &= w^T M w \Big|_r^x - w^T M w \Big|_l^x \\ &= w^T M^- w \Big|_r^x + g_r^T M^+ g_r - w^T M^+ w \Big|_l^x - g_l^T M^- g_l, \end{aligned}$$

which contains terms that are either bounded by boundary data or negative semidefinite. Therefore, (19) is a set of strongly well-posed boundary conditions.

We prove that the following SBP-SAT approximation of (12) is stable.

$$\begin{cases} u_t = -3\bar{v} D_1 v \\ -H^{-1} \begin{bmatrix} e_m & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} M_r^+(w_m - g_r) + H^{-1} \begin{bmatrix} e_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} M_l^-(w_1 - g_l) \\ v_t = -\bar{v} D_1 u - 2\bar{u} D_1 v - 2D_3 v \\ -H^{-1} \begin{bmatrix} \mathbf{0} & e_m & d_{1;m}^T & d_{2;m}^T \end{bmatrix} M_r^+(w_m - g_r) + H^{-1} \begin{bmatrix} \mathbf{0} & e_1 & d_{1;1}^T & d_{2;1}^T \end{bmatrix} M_l^-(w_1 - g_l), \end{cases} \quad (20)$$

where  $\mathbf{0}$  is an  $m \times 1$  vector of zeros;  $u, v$  are the discrete variables approximating the continuous ones (here we use the same notation);  $\bar{u}, \bar{v}$  are the diagonal matrices of which the diagonals are

$$u, v, \text{ respectively; } w_1 = \begin{bmatrix} u_1 \\ v_1 \\ d_{1;1} v \\ d_{2;1} v \end{bmatrix}, w_m = \begin{bmatrix} u_m \\ v_m \\ d_{1;m} v \\ d_{2;m} v \end{bmatrix}.$$



Using the energy method,

$$u^T H u_t = -3u^T \bar{v} \left( Q + \frac{1}{2} B \right) v - [u_m, 0, 0, 0] M_r^+ (w_m - g_r) + [u_1, 0, 0, 0] M_l^- (w_1 - g_l).$$

Because  $M^+, M^-$  are symmetric, we have

$$u_t^T H u = -3v^T \left( Q^T + \frac{1}{2} B \right) \bar{v} u - (w_m - g_r)^T M_r^+ \begin{bmatrix} u_m \\ 0 \\ 0 \\ 0 \end{bmatrix} + (w_1 - g_l)^T M_l^- \begin{bmatrix} u_1 \\ 0 \\ 0 \\ 0 \end{bmatrix};$$

$$\begin{aligned} v^T H v_t &= -v^T \bar{v} \left( Q + \frac{1}{2} B \right) u - 2v^T \bar{u} \left( Q + \frac{1}{2} B \right) v - 2v^T H D_3 v \\ &\quad - [0, v_m, d_{1;m} v, d_{2;m} v] M_r^+ (w_m - g_r) + [0, v_1, d_{1;1} v, d_{2;1} v] M_l^- (w_1 - g_l); \end{aligned}$$

$$\begin{aligned} v_t^T H v &= -u^T \left( Q^T + \frac{1}{2} B \right) \bar{v} v - 2v^T \left( Q^T + \frac{1}{2} B \right) \bar{u} v - 2(D_3 v)^T H v \\ &\quad - (w_m - g_r)^T M_r^+ \begin{bmatrix} 0 \\ v_m \\ d_{1;m} v \\ d_{2;m} v \end{bmatrix} + (w_1 - g_l)^T M_l^- \begin{bmatrix} 0 \\ v_1 \\ d_{1;1} v \\ d_{2;1} v \end{bmatrix}. \end{aligned}$$

Therefore, we have the semi-discrete energy estimate

$$\begin{aligned} \frac{d}{dt} (\|u\|_H^2 + \|v\|_H^2) &= -5u^T \bar{v} \left( Q + \frac{1}{2} B \right) v - 5v^T \left( Q^T + \frac{1}{2} B \right) \bar{v} u \\ &\quad - v^T \bar{v} \left( Q + \frac{1}{2} B \right) u - u^T \left( Q^T + \frac{1}{2} B \right) \bar{v} v \\ &\quad - 2v^T H D_3 v - 2(D_3 v)^T H v \\ &\quad - w_m^T M_r^+ (w_m - g_r) + w_1^T M_l^- (w_1 - g_l) \\ &\quad - (w_m - g_r)^T M_r^+ w_m + (w_1 - g_l)^T M_l^- w_1 \\ &= \text{BT} + \text{RT}, \end{aligned}$$

where the boundary term is then,

$$\begin{aligned} \text{BT} &= -6u_m v_m^2 + 6u_1 v_1^2 - 2(d_{1;1} v)^2 + 2(d_{1;m} v)^2 + 4v_1 d_{2;1} v - 4v_m d_{2;m} v \\ &\quad - w_m^T M_r^+ (w_m - g_r) + w_1^T M_l^- (w_1 - g_l) \\ &\quad - (w_m - g_r)^T M_r^+ w_m + (w_1 - g_l)^T M_l^- w_1. \end{aligned}$$

We can rewrite it in the full matrix form

$$\begin{aligned} \text{BT} &= w_m^T M_r w_m - w_1^T M_l w_1 \\ &\quad - w_m^T M_r^+ (w_m - g_r) + w_1^T M_l^- (w_1 - g_l) \\ &\quad - (w_m - g_r)^T M_r^+ w_m + (w_1 - g_l)^T M_l^- w_1 \\ &= w_m^T M_r^- w_m - w_1^T M_l^+ w_1 + g_r^T M_r^+ g_r - g_l^T M_l^- g_l \\ &\quad - (w_m - g_r)^T M_r^+ (w_m - g_r) + (w_1 - g_l)^T M_l^- (w_1 - g_l). \end{aligned}$$

which mimics the continuous energy estimate with additional damping terms.

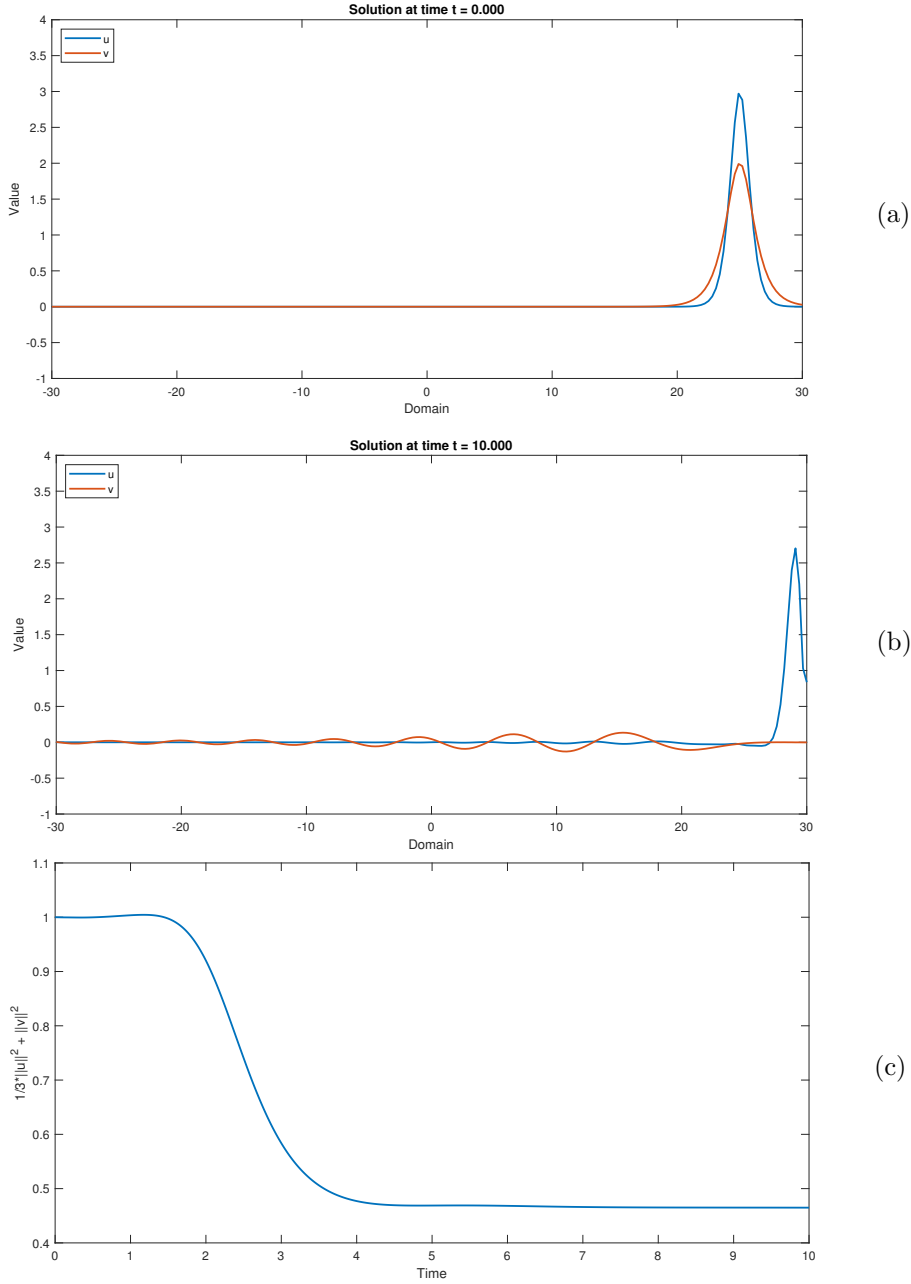


Figure 6: Boundary interaction with characteristic boundary conditions at time  $t = 0$  and  $t = 10$ ,  $m = 201$ .

Two snapshots at different time points of the boundary interaction with the new set of boundary conditions and the total energy over the course of interaction are shown in Figure 6. The observed behaviour differs from the Dirichlet boundary conditions. Although a stationary pulse in variable  $u$  is still strangely left on the right boundary. However, from the energy plot (Figure 6(c)), it can be seen that the energy almost strictly decays over time. It can be explained by the new set of boundary conditions that introduces more damping on the boundaries.

A convergence analysis is performed on the domain  $[x_l, x_r] = [-30, 30]$  with  $m$  space grid points, the wave speed  $c = 2$ , time step size  $\Delta t = \beta h^3$  where  $h = (x_r - x_l)/(m - 1)$ ,  $\beta = 0.01, 0.001, 0.0005$  for  $2^{nd}, 4^{th}, 6^{th}$  order operators, respectively. The result is shown in Table 5.

$m$	error <sup>(2nd)</sup>	$q^{(2nd)}$	error <sup>(4th)</sup>	$q^{(4th)}$	error <sup>(6th)</sup>	$q^{(6th)}$
101	0.2422	-	0.0556	-	0.0203	-
151	0.1125	1.9071	0.0114	3.9397	0.0022	5.5336
201	0.0637	1.9889	0.0037	3.9716	0.0004	5.8077

Table 5:  $L_2$ -norm of the error and convergence rate for the DSW problem with the set of boundary conditions (19) by different accuracy orders of SBP operators and number of space grid points. Solutions are examined at  $t = 5$ .

## 4 Conclusion

In this project, we study the concepts of soliton and two PDE-models (Korteweg-de Vries equation and the coupled Drinfeld-Sokolov-Wilson Equation) that can be used to describe such phenomenon. The SBP operators for finite difference method are applied to solve the periodic problem. We have analysed the involvement of boundary conditions by deriving two different sets of well-posed boundary conditions for each equation. The implementation of those boundary conditions utilizes the SAT technique, of which accuracy and stability is shown in the numerical result by different orders of accuracy. For the KdV equation, we show nonlinear stability using the energy method and skew-symmetric splitting. However, for the DSW equation, we could only show linear stability in a special norm. The numerical solution converges to the exact solution with the expected convergence rate. The boundary interaction is illustrated in figures. The numerical study also strengthens the motivation for higher order accurate SBP-SAT.

Future work may consist of applying optimized time-stepping to the discrete approximations. One could also investigate the possibility of obtaining a nonlinear energy estimate for the DSW equation.

## References

- [1] Almqvist, M., Mattsson, K., and Edvinsson, T. (2014). High-fidelity numerical solution of the time-dependent Dirac equation. *Journal of Computational physics*, 262, 86-103.
- [2] Brauer, K. (2000). *The Korteweg-de Vries equation: history, exact solutions, and graphical representation*. University of Osnabrück/Germany1.
- [3] Carpenter, M. H., Gottlieb, D., and Abarbanel, S. (1994). Time-stable boundary conditions for finite-difference schemes solving hyperbolic systems: methodology and application to high-order compact schemes. *Journal of Computational Physics*, 111(2), 220-236.
- [4] Drinfel'd, V. G., & Sokolov, V. V. (1985). Lie algebras and equations of Korteweg-de Vries type. *Journal of Soviet mathematics*, 30(2), 1975-2036.
- [5] Gustafsson, B., Kreiss, H. O., & Olinger, J. (1995). *Time dependent problems and difference methods* (Vol. 24). John Wiley & Sons.
- [6] Hirota, R., Grammaticos, B., & Ramani, A. (1986). Soliton structure of the Drinfeld-Sokolov-Wilson equation. *Journal of mathematical physics*, 27(6), 1499-1505.
- [7] Horne, R. L. (2002). *A (Very) Brief Introduction to Soliton Theory in a class of Nonlinear PDEs*. *Mathematical Sciences Proceedings*.
- [8] Kreiss, H. O., and Scherer, G. (1974). Finite element and finite difference methods for hyperbolic partial differential equations. In *Mathematical aspects of finite elements in partial differential equations* (pp. 195-212).
- [9] Marchant, T. R. (2004). Asymptotic solitons for a third-order Korteweg-de Vries equation. *Chaos, Solitons & Fractals*, 22(2), 261-270.

- [10] Mattsson, K. (2014). Diagonal-norm summation by parts operators for finite difference approximations of third and fourth derivatives. *Journal of Computational Physics*, 274, 432-454.
- [11] Mattsson, K., and Werpers, J. (2016). High-fidelity numerical simulation of solitons in the nerve axon. *Journal of Computational Physics*, 305, 793-816.
- [12] Svård, M. and Nordström, J. (2014). Review of summation-by-parts schemes for initial-boundary-value problems. *Journal of Computational Physics*, 268, 17-38. Chicago
- [13] Wilson, G. (1982). The affine Lie algebra  $C_2^{(1)}$  and an equation of Hirota and Satsuma. *Physics Letters A*, 89(7), 332-334.
- [14] Zhang, W. M. (2011). Solitary solutions and singular periodic solutions of the Drinfeld-Sokolov-Wilson Equation by variational approach. *Appl Math Sci*, 5(38), 1887-1894.