Project: Time-step analysis of methods for the advection equation

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Overview

This project is about the stability of different high order numerical methods for the advection equation on a $1D$ periodic[1] spatial domain:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, & x \in [0,1]\,, t > 0, \\ u(x+1,t) = u(x,t), & x \in [0,1)\,, t \geq 0, \\ u(x,0) = u_0(x), & x \in [0,1)\,. \end{cases}$$

For simplicity time will be discretized separately to space and integrated over with an explicit time-stepping method, such as forward Euler or fourth order Runge-Kutta (RK4). The two main methods of spatial discretization to be considered are finite difference and continuous (Lagrange) finite element [1] schemes. Prior knowledge of these is not required, and part of the project is to learn about these methods via a literature study. Some simple stabilization techniques, such as artificial viscosity methods for finite elements, should also be considered.

Project outline

The project will consist of two sections; an brief introduction and the main investigative work. The introduction is primarily to build familiarity with the methods to be used, and should contain the following two parts:

1. Von Neumann stability analysis for the central difference discretization and piecewise linear finite elements, and an implementation of the schemes with RK4 time-stepping to check results.
2. A short literature study covering higher order finite difference methods, higher order Lagrange finite elements (sometimes referred to as spectral elements), stabilization techniques such as artificial viscosity methods, stability analysis and methods of eigenvalue computations.

Once the first part of the introduction is complete the main section should begin. Investigation should address the following questions for both stabilized and unstabilized techniques. All questions should be investigated by computational experiments, and where possible by mathematical analysis as well.

1. For higher order Lagrange finite elements, does the maximum stable time-step depend on how the polynomial interpolation points[2] are chosen?
2. How does the maximum stable time-step behave as the order of the methods increase?
3. In cases where a stabilization technique is not necessary to obtain a stable solution, what is the effect on the maximum time-step of applying the stabilization?

The project may also be extended by considering other methods, such as discontinuous Galerkin [2] finite elements. However the above sections should be completed beforehand.

[1] By making the spatial domain periodic we avoid stability constraints arising from how boundary conditions are implemented. This also makes von Neumann stability analysis (discrete Fourier transform) simpler to use.

[2] The points at which only one basis polynomial on a given element is non-zero. Common choices for these points are to evenly distribute them across the element, or use a set of Gauss-Lobatto [3] quadrature points.

Prerequisites

Students should have taken the course Scientific Computing III (course code 1TD397), or have a similar level of knowledge of numerical methods. In particular they should understand the principles of stability analysis, and be familiar with finite element and finite difference methods for time-dependent problems. Students should also be interested in working with the mathematical theory of numerical analysis, as this project is focused towards a theoretical result rather than implementing a numerical solver for a complex problem.

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References

[1] P. G. Ciarlet, *Introduction to the finite element method*, vol. 4, pp. 36–109. ScienceDirect, 1978.

[2] M. G. Larson and F. Bengzon, *The Finite Element Method: Theory, Implementation, and Applications.* Springer, 2013.

[3] N. Kovvali, *Theory and Applications of Gaussian Quadrature Methods.* Morgan and Claypool, 2011.