



Intro. Computer Control Systems: F10

Sensitivity and robustness

Dave Zachariah

Dept. Information Technology, Div. Systems and Control



F9: Quiz!

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- a the states can be estimated arbitrarily well \uparrow
- b the states can be controlled arbitrarily well \uparrow
- c the system is also stable \downarrow

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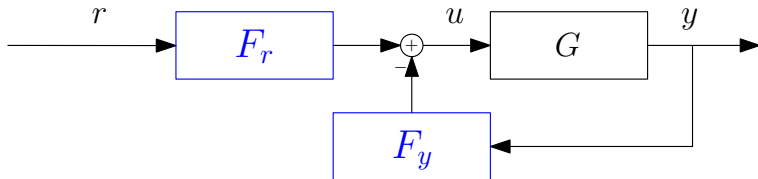
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- 3) The transfer function for a control system with **estimated states**
 - a is different from that of control system with known states \uparrow
 - b is the same as that of control system with known states \uparrow
 - c is real-valued \downarrow

Control system with disturbances and noise

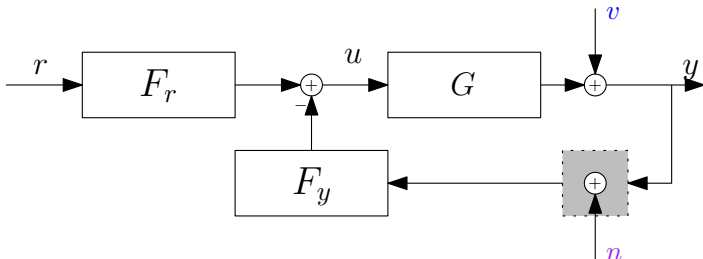
Using general linear feedback



How will the control system cope with *unknown* disturbances and noise?

Control system with disturbances and noise

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[Board: the closed-loop system with $V(s)$ and $N(s)$]

Sensitivity functions

- ▶ Open-loop system: $G_o(s) \triangleq F_y(s)G(s)$
- ▶ **Sensitivity function:**

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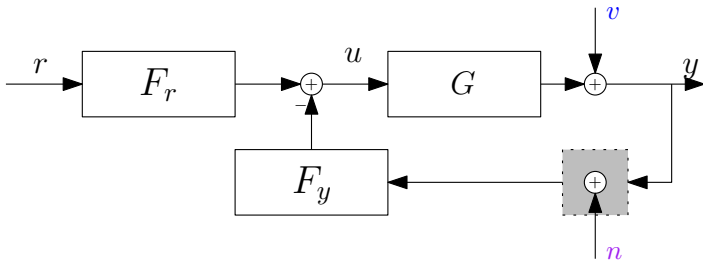
$$T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)}$$

- ▶ Consequence:

$$S(s) + T(s) \equiv 1, \quad \forall s$$

- ▶ $S(s)$ and $T(s)$ affected by controller $F_y(s)$.

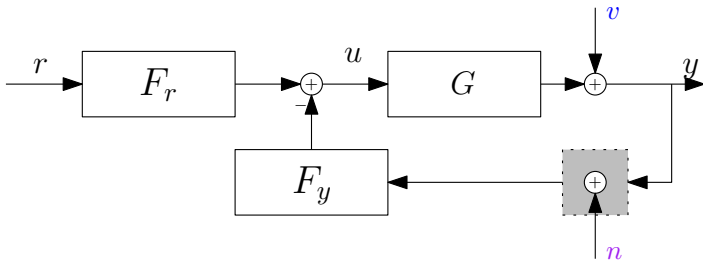
Closed-loop system and the sensitivity functions



- Closed-loop system:

$$Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s)$$

Closed-loop system and the sensitivity functions

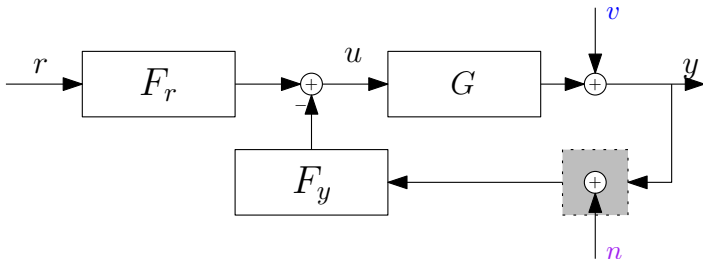


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- ...but *impossible* since

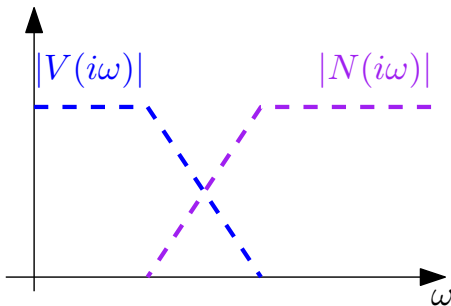
$$|S(i\omega)| + |T(i\omega)| \geq |S(i\omega) + T(i\omega)| \equiv 1$$

Closed-loop system and sensitivity functions

Design trade-off

Example:

- ▶ **Disturbance** $v(t)$ with energy at low frequencies
- ▶ **Noise** $n(t)$ with energy at high frequencies

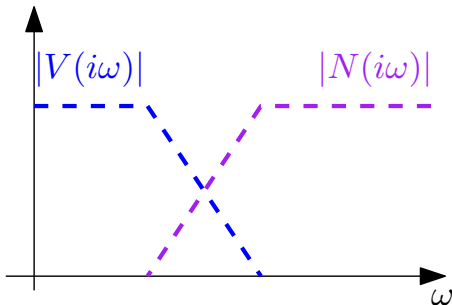


Closed-loop system and sensitivity functions

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Example:

- ▶ **Disturbance** $v(t)$ with energy at low frequencies
- ▶ **Noise** $n(t)$ with energy at high frequencies



Typical design trade-off is then:

- ▶ *low* ω : $|S(i\omega)| \ll 1$ to suppress $V(i\omega)$.
- ▶ *high* ω : $|T(i\omega)| \ll 1$ to suppress $N(i\omega)$.

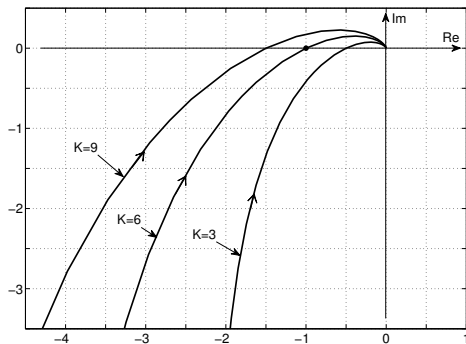
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Design trade-off

Simultaneously we want Nyquist contour

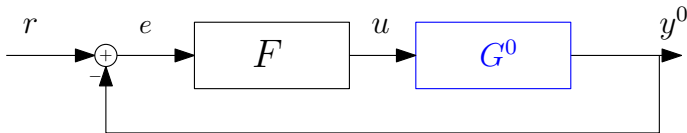
$$G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)}, \quad -\infty \leq \omega \leq \infty$$

far from -1 . (Cf. F6 and F7.)



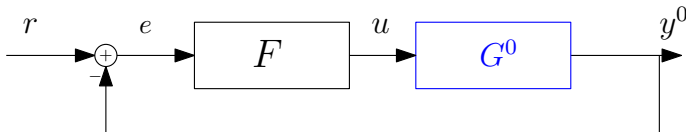
Control systems with model errors

All *models* are *approximate*



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- Assume that the **real system** can be written as

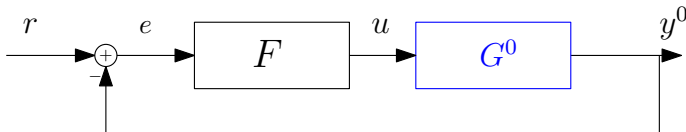
$$G^0(s) = G(s)(1 + \Delta_G(s))$$

- The **relative model error** for $G(s)$:

$$\Delta_G(s) = \frac{G^0(s) - G(s)}{G(s)}$$

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- How is stability affected by unknown error $\Delta_G(s)$?

Model errors and stability

Assume:

1. Controller $F(s)$ stabilizes the *assumed* system $G(s)$
2. $G(s)$ and $G^0(s)$ have same number of poles in right half-plane.
3. Open-loop: $F(s)G(s) \rightarrow 0$ and $F(s)G^0(s) \rightarrow 0$ where $|s| \rightarrow \infty$

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(Result 6.2) Robustness criterium

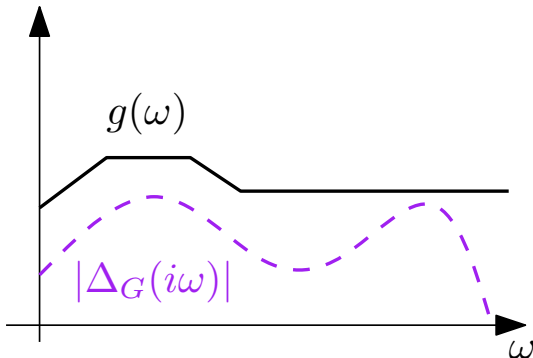
If the assumptions are valid and complementary sensitivity function fulfills

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad -\infty \leq \omega \leq \infty$$

\Rightarrow the *real* closed-loop system is also **stable**!

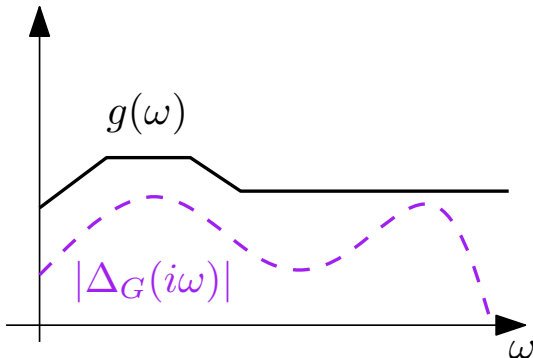
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$$|T(i\omega)| < \frac{1}{g(\omega)} < \frac{1}{|\Delta_G(i\omega)|}$$

\Rightarrow real closed-loop system is also **stable**

Summary and recap

- ▶ Sensitivity with respect to disturbances and noise
- ▶ Sensitivity functions and their impact on control
- ▶ Robustness with respect to model errors
- ▶ Robustness criterion in the frequency domain