



Intro. Computer Control Systems: F11

Discrete-time control

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F10: Quiz!

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- 1) Sensitivity function $S(s)$
 - a does not affect control \uparrow
 - b amplifies output disturbances \uparrow
 - c is stable \downarrow

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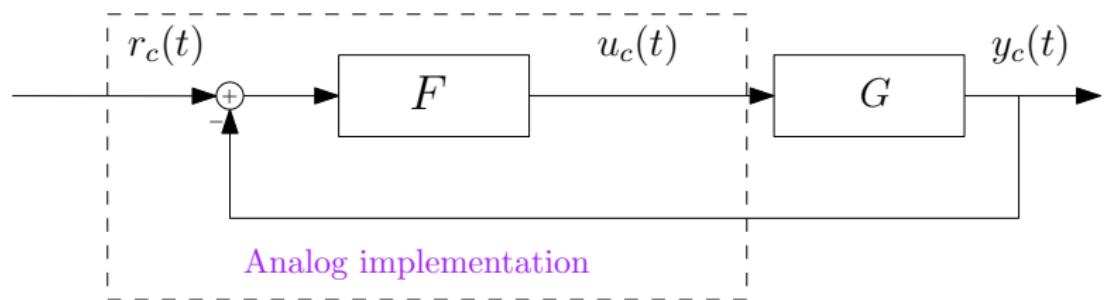
- a sum to 1 \uparrow
- b sum to $G_o(s)$ \uparrow
- c sum to a complex-valued number \downarrow

3) If $|T(i\omega)|$ is below $1/|\Delta_G(i\omega)|$

- a the unknown system $G^0(s)$ is stable \uparrow
- b the closed-loop system can be stabilized despite unknown system $G^0(s)$ \uparrow
- c the output becomes real-valued \downarrow

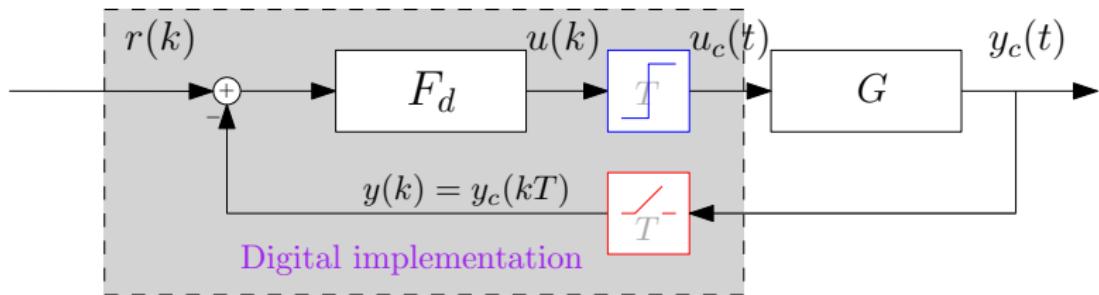
Digital systems and discrete-time clocks

Example: Simple linear feedback control system



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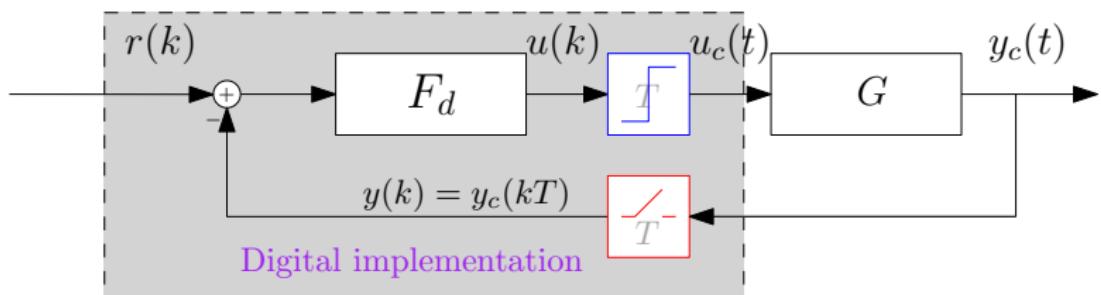


Note: discrete-time signals with sample interval T

- ▶ **A/D:** Sampling feedback signal $y(k) = y_c(kT)$
- ▶ **D/A:** Generate continuous-time input signal $u(t)$

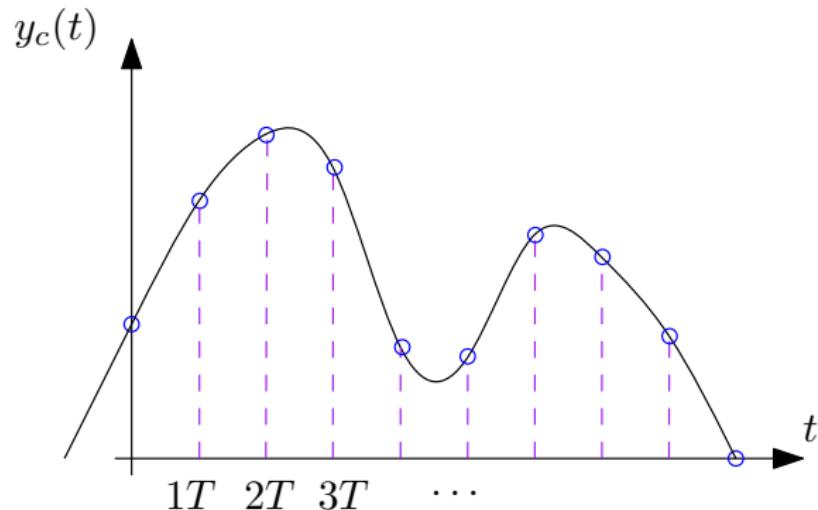
Digital systems and discrete-time clocks

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A/D: Sampling feedback signal $y(t) = y_c(kT)$

Sampling the feedback signal



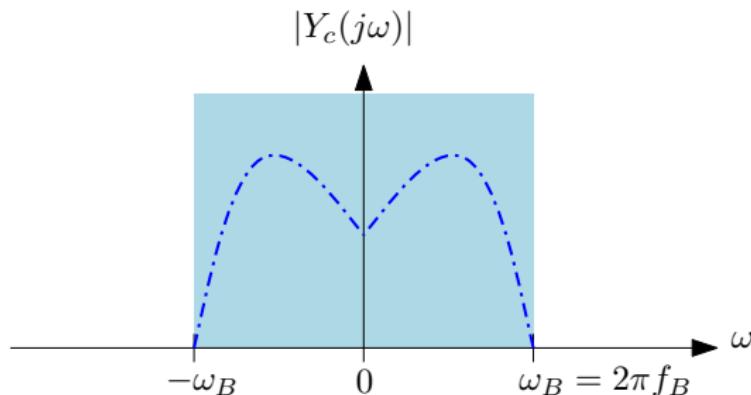
$$y(k) = y_c(kT), \quad k = 0, 1, \dots$$

Sampling the feedback signal

Nyquist sampling theorem:

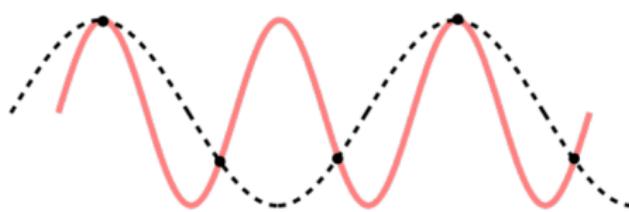
If $y_c(t)$ is bandlimited to $\pm\omega_B = \pm 2\pi f_B$ \Rightarrow it can be recovered exactly from $y_c(kT)$ when sampling frequency

$$f_s = \frac{1}{T} \geq 2f_B$$



Sampling the feedback signal

If $T > \frac{1}{2f_B}$ we may not resolve $y(t)$ uniquely resulting in distortion/‘aliasing’.



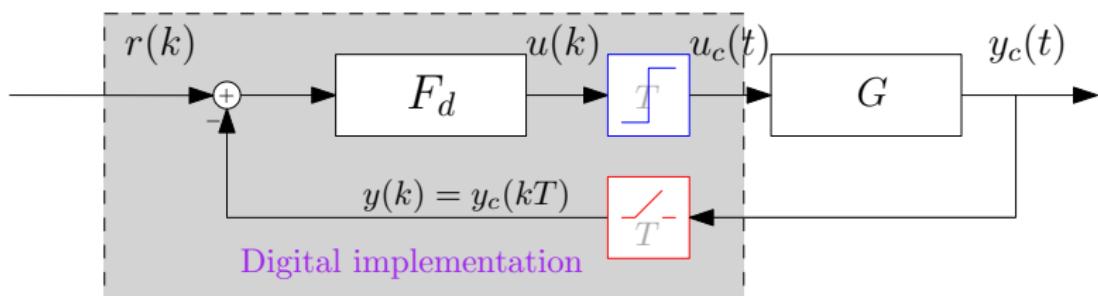
High frequency components $f > 1/(2T)$ may appear at low frequencies.

Practical solution:

1. Low-pass filter $y_c(t)$ with $L(s)$: output $y_L(t)$ bandlimited to $f_B = 1/(2T)$.
2. Then sample filtered signal $y(k) = y_L(kT)$.

Digital systems and discrete-time clocks

Example: Simple linear feedback control system

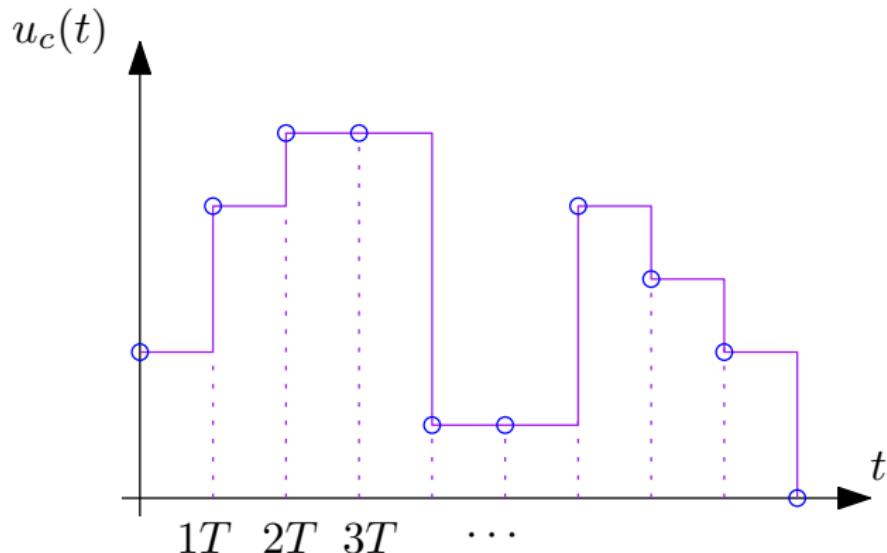


D/A: Generate continuous-time input signal $u(t)$

Generate input signal using zero-order hold

Zero-order hold: continuous-time input is *piecewise constant* for $k = 0, 1, 2, \dots$

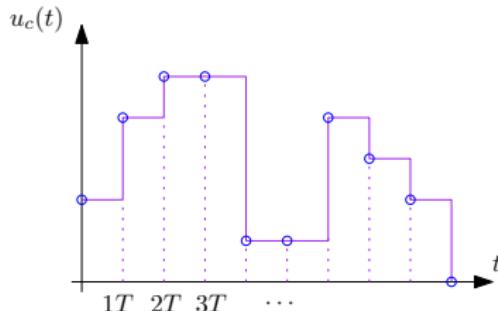
$$u_c(t) = u_c(kT), \quad kT \leq t < (k+1)T$$



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How do the states $x_c(t)$ evolve with input $u_c(t)$ above,

$$\dot{x}_c(t) = Ax_c(t) + B\textcolor{blue}{u}_c(t),$$

starting at $x_c(kT)$?

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$$u_c(t) = u_c(kT), \quad kT \leq t < (k+1)T$$

Starting at $x_c(kT)$, general solution:

$$x_c(t) = e^{A(t-kT)} x_c(kT) + \int_{kT}^t e^{A(t-\tau)} B u_c(\tau) d\tau$$

at $t = (k+1)T$

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at $t = (k+1)T$:

$$x_c((k+1)T) = e^{AT} x_c(kT) + \left(\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} d\tau B \right) u_c(kT)$$

Discrete-time system model

State-space descriptions

Instead of

$$x_c((k+1)T) = e^{AT} x_c(kT) + \left(\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} d\tau B \right) u_c(kT),$$

discrete-time

$$x(k+1) = Fx(k) + Gu(k)$$

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$$x(k+1) = Fx(k) + Gu(k)$$

where

$$F = e^{AT}$$

$$G = \int_{\tau=0}^T e^{A\tau} d\tau B = [\text{if } A^{-1} \text{ exists}] = A^{-1}(e^{AT} - I)B$$

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and matrix exponential as before

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

Build intuition from simple systems

Discrete-time state-space description

Example: Continuous-time system with **zero-order hold input**

$$\dot{x}_c(t) = Ax_c(t) + Bu_c(t)$$

$$y_c(t) = Cx_c(t),$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 1]$$

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[Board: derive discrete-time state-space description]

$$\boxed{\begin{aligned}x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k)\end{aligned}}$$

Note: Easy to simulate in computer!

Discrete-time system model

State-space descriptions

General solution to $\dot{x}_c(t) = Ax_c(t) + Bu_c(t)$:

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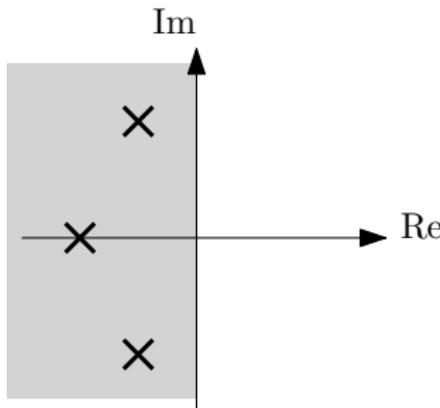
If eigenvalues of F : $|p| < 1 \Rightarrow \text{stable}$

Discrete-time LTI systems

Eigenvalues/poles

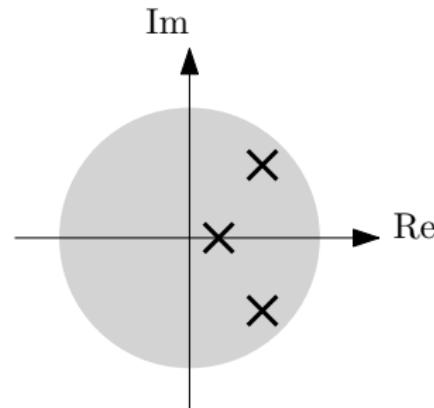
Stability: left half-plane vs. unit circle

Continuous-time LTI



$$\text{Re}\{p\} < 0$$

Discrete-time LTI



$$|p| < 1$$

Discrete-time LTI systems

Eigenvalues/poles and sampling

Eigenvalues/poles of sampled system

System $\dot{x}_c(t) = Ax_c(t) + Bu_c(t)$ with eigenvalues p_1, \dots, p_n .

Using zero-order hold input, sampled system

$$x(k+1) = Fx(k) + Gu(k)$$

⇒ eigenvalues of $F = e^{\textcolor{blue}{A}T}$:

$$\tilde{p}_1 = e^{\textcolor{red}{p_1}T}, \dots, \tilde{p}_n = e^{\textcolor{red}{p_n}T}$$

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Note: Large T /**slower** sampling, increases distance $|\tilde{p}| = |e^{pT}|$!

Discrete-time controllers

- ▶ Ideal PID controller:

$$u(k) = K_p e(k) + K_i \underbrace{T \sum_{n=0}^k e(n)}_{\simeq \int_{\tau=0}^t e_c(\tau) d\tau} + K_d \underbrace{\frac{(e(k) - e(k-1))}{T}}_{\simeq \frac{de_c(t)}{dt}}$$

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$$u(k) = -Lx(k) + \ell_0 r(k)$$

Plug back into state-space equations to obtain closed-loop system

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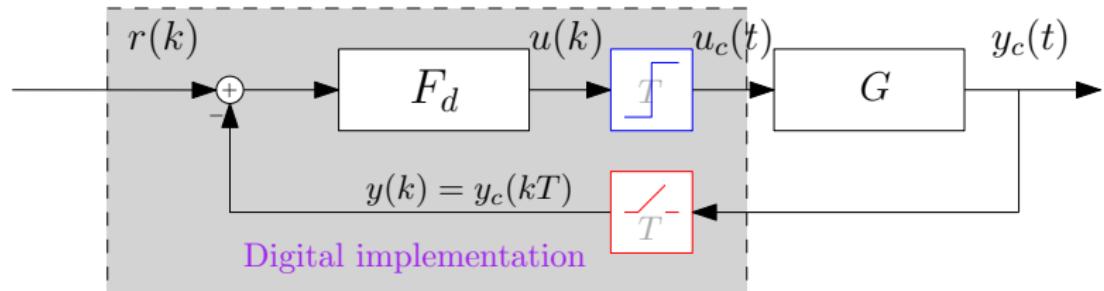
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More advanced controllers in Automatic Control III!

Discrete-time controllers



More advanced controllers in Automatic Control III!

Summary and recap

- ▶ Sampling output and Nyquist sampling theorem
- ▶ Zero-order hold input and state-space form
- ▶ Discrete-time LTI eigenvalues/poles