



Intro. Computer Control Systems: F2

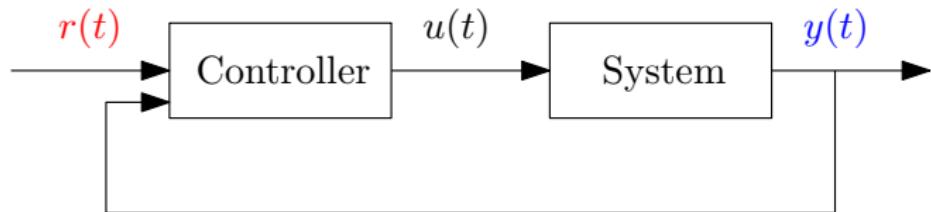
Transfer function, poles and stability

Dave Zachariah

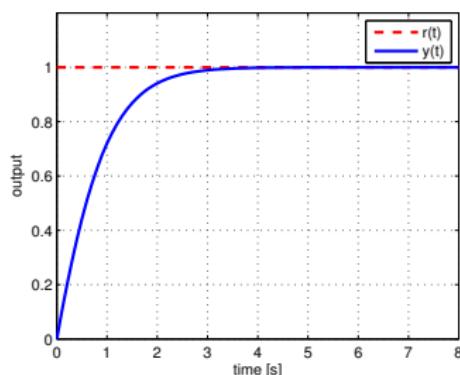
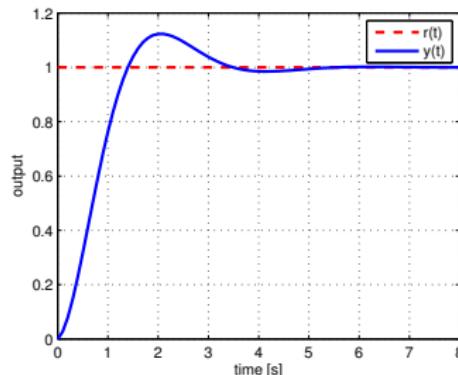
Dept. Information Technology, Div. Systems and Control

F1: Quiz!

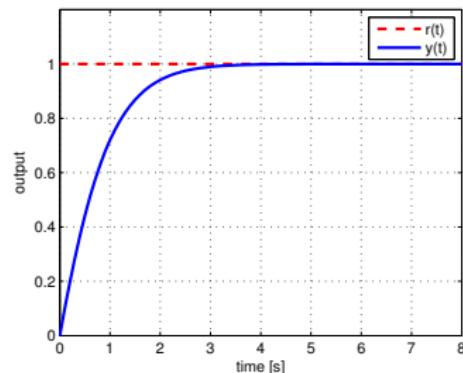
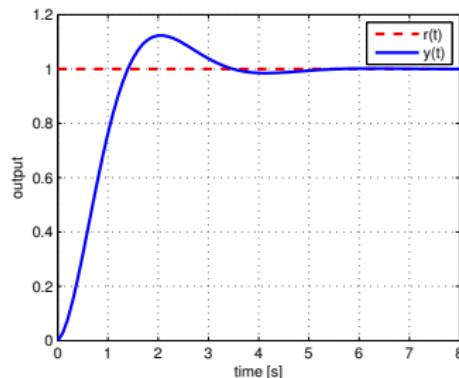
Suppose a control system



comes with two different settings (a) and (b).



F1: Quiz!

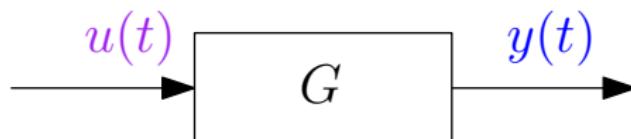


- 1) Which setting of the controller is intuitively better?
- a Setting to the left ↑
 - b Setting to the right ↑
 - c They are equally good ↓



Linear system models

Linear time-invariant models are useful and sufficiently accurate in many control applications.



Linear ODE:s are *one possible* input-output description, i.e. of G :

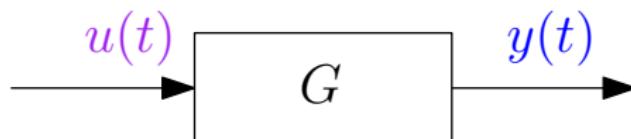
$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_ny = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_mu$$

with initial conditions.



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Rarely practical in analysis or design for control!

Laplace transform

Used as tool to *solve* and *analyze* linear ODE:s

► **Notation:**

$$y(t) \quad \xleftrightarrow{\mathcal{L}} \quad \mathcal{L}[y(t)] = Y(s)$$

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► **Notation:**

$$y(t) \quad \xleftrightarrow{\mathcal{L}} \quad \mathcal{L}[y(t)] = Y(s)$$

► **Definition:**

$$Y(s) = \mathcal{L}[y(t)] = \int_0^{\infty} y(t)e^{-st}dt, \quad s \in \mathbb{C}$$

Inverse transform:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi i} \int_{\mathbb{C}} Y(s)e^{st}ds, \quad s \in \mathbb{C}$$

Note that s and $Y(s)$ are complex-valued!

Important properties

linearity: $y(t) = \alpha x(t) + \beta z(t) \xleftrightarrow{\mathcal{L}} Y(s) = \alpha X(s) + \beta Z(s)$

derivatives: $\frac{dy}{dt} \xleftrightarrow{\mathcal{L}} sY(s) - y(0)$

$$\frac{d^2y}{dt^2} \xleftrightarrow{\mathcal{L}} s^2Y(s) - sy(0) - \dot{y}(0)$$

⋮

integral: $\int_0^t y(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}Y(s)$

convolution: $\int_0^t x(\tau)z(t-\tau)d\tau \xleftrightarrow{\mathcal{L}} X(s)Z(s)$

final-value thm.*: $\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$

Solving linear ODE with \mathcal{L}

Exemple: solve **output** $y(t)$:

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = 4\frac{d}{dt}u + 5u, \quad u(t), y(0), \dot{y}(0) \text{ given}$$

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Laplace transform \implies

$$\begin{aligned}\text{LHS} &= s^2Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + 3Y(s) \\ &= (s^2 + 2s + 3)Y(s) - (s + 2)y(0) - \dot{y}(0)\end{aligned}$$

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Set LHS = RHS and solve for $Y(s)$: \Rightarrow

$$Y(s) = \frac{4s + 5}{s^2 + 2s + 3}U(s) + \frac{s + 2}{s^2 + 2s + 3}y(0) + \frac{1}{s^2 + 2s + 3}(\dot{y}(0) - 4u(0))$$

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$U(s) = \mathcal{L}[u(t)]$, $u(0)$, $y(0)$ and $\dot{y}(0)$ are given \Rightarrow

Use \mathcal{L}^{-1} -transform (table) to compute $y(t) = \mathcal{L}^{-1}[y(t)]$.

Transfer function

Study effect of input u — **Assuming** initial values
 $y(0) = \dot{y}(0) = \dots = 0$ and $u(0) = \dot{u}(0) = \dots = 0$:

$$\Rightarrow Y(s) = \frac{4s + 5}{s^2 + 2s + 3} U(s) = G(s)U(s)$$

- ▶ $G(s)$ is the system **transfer function** $u \rightarrow y$.
- ▶ $Y(s) = G(s)U(s)$ is a model of the *relation* between the system input u and output y .

Transfer function

- ▶ A system described by the linear ODE

$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

with **initial values 0**.

- ▶ Laplace transform of both sides:

$$(s^n + \cdots + a_{n-1}s + a_n)Y(s) = (b_0s^m + \cdots + b_{m-1}s + b_m)U(s)$$

Transfer function

- ▶ A system described by the linear ODE

$$\frac{d^n}{dt^n}y + \cdots + \color{blue}{a_{n-1}}\frac{d}{dt}y + \color{blue}{a_n}y = \color{magenta}{b_0}\frac{d^m}{dt^m}u + \cdots + \color{magenta}{b_{m-1}}\frac{d}{dt}u + \color{magenta}{b_m}u$$

with **initial values 0**.

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$$(s^n + \cdots + \color{blue}{a_{n-1}}s + \color{blue}{a_n})Y(s) = (\color{magenta}{b_0}s^m + \cdots + \color{magenta}{b_{m-1}}s + \color{magenta}{b_m})U(s)$$

- ▶ System transfer function is a *rational* function:

$$G(s) = \frac{\color{magenta}{b_0}s^m + \cdots + \color{magenta}{b_m}}{s^n + \color{blue}{a_1}s^{n-1} + \cdots + \color{blue}{a_n}}$$

Note that s and $G(s)$ are complex-valued!

Weighting function/impulse response

A system $Y(s) = G(s)U(s)$ (at rest $t = 0$) yields

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \int_0^t g(\tau)u(t - \tau)d\tau,$$

i.e. a *convolution* between $u(t)$ and

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

denoted the system **weighting function**.

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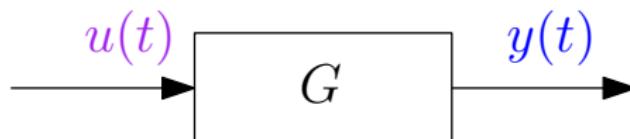
Suppose input $u(t) = \delta(t)$ = (Dirac)pulse, then output

$$y(t) = \int_0^t g(\tau)\delta(t - \tau)d\tau = g(t).$$

Hence $g(t)$ is called the system **impulse response**.

Poles and zeros

Characterizing system behaviour

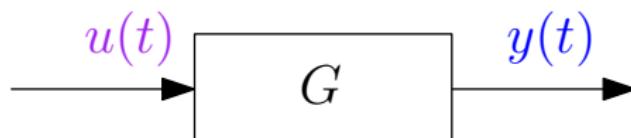


System with transfer function $G(s)$

- ▶ **Zeros:** s' is a *zero*, if $G(s') = 0$.
- ▶ **Poles:** s' is a *pole*, if $G(s')$ is a singularity, that is, $G(s') = \pm\infty$.

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- ▶ **Poles:** s' is a *pole*, if $G(s')$ is a singularity, that is, $G(s') = \pm\infty$.
- ▶ If $G(s) = \frac{B(s)}{A(s)}$ is a rational function one obtains
 - ▶ zeros from the roots to $B(s) = 0$,
 - ▶ och poles from the roots to $A(s) = 0$.

Poles and solution to linear ODE:s

Characterizing system behaviour

- ▶ Assume system $Y(s) = G(s)U(s)$, where $G(s) = \frac{B(s)}{A(s)}$.
- ▶ We want

$$y(t) = \int_0^t g(\tau)u(t - \tau)d\tau$$

where $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$.

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where $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$.

- ▶ Denominator always factorize with roots/poles:

$$\begin{aligned} A(s) &= s^n + a_1s^{n-1} + \cdots + a_n \\ &= (s + \sigma_1) \cdots ((s + \sigma_j)^2 + \omega_j^2) \cdots \end{aligned}$$

where poles are either

- ▶ real-valued: $-\sigma_1, \dots$
- ▶ complex-conjugated: $-\sigma_j \pm i\omega_j, \dots$

Poles and solution to linear ODE:s

Characterizing system behaviour

- ▶ Insert $A(s)$ use partial-fraction decomposition

$$G(s) = \frac{B(s)}{A(s)} = \frac{\beta_1}{s + \sigma_1} + \cdots + \frac{B_j(s)}{(s + \sigma_j)^2 + \omega_j^2} + \cdots$$

- ▶ So that

$$y(t) = \int_0^t g(\tau)u(t - \tau)d\tau$$

using inverse transform (table) gives

$$g(t) = \beta_1 e^{-\sigma_1 t} + \cdots + b_j e^{-\sigma_j t} \sin(\omega_j t + \varphi_j) + \cdots$$

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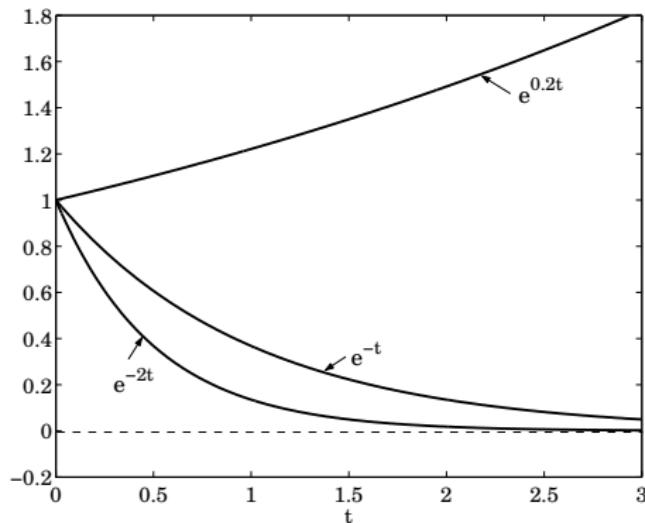
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Linear combination of exponential functions

Poles and solution to linear ODE:s

Characterizing system behaviour

Solutions to linear ODE:s are equivalent to linear combinations of exponential functions



Real-parts of poles play an important role

Stability

Characterizing system behaviour

Definition:

A system $Y(s) = G(s)U(s)$ is **input-output stable** if every bounded input $u(t)$ gives a bounded output $y(t)$.

Signal $x(t)$ bounded $\Leftrightarrow |x(t)| \leq K$ for some K .

[Board: bounded impulse response + real-part of poles]

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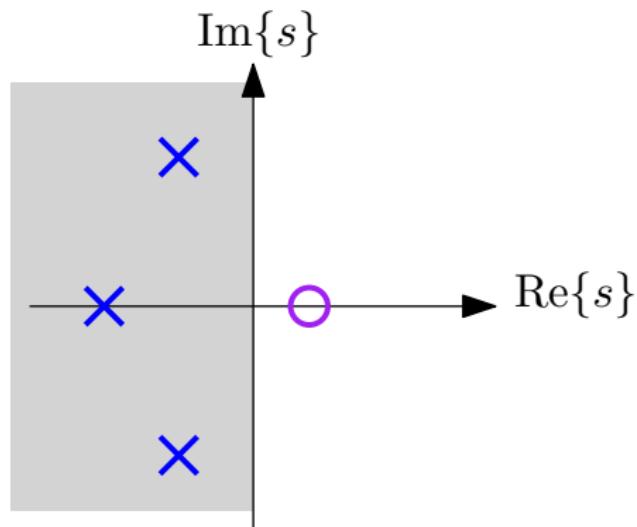
Result:

Assume $G(s) = B(s)/A(s)$ where order in denominator \geq numerator och poles p_1, p_2, \dots, p_n

System $Y(s) = G(s)U(s)$ is **input-output stable** $\Leftrightarrow \text{Re}\{p_i\} < 0$

Graphical representation of poles and zeros

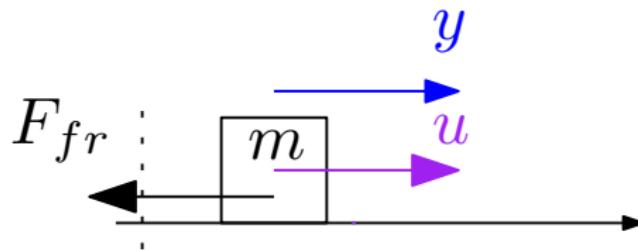
Characterizing system behaviour



$G(s) = \frac{B(s)}{A(s)}$ stable \Leftrightarrow poles lie in left-halfplane

Build intuition from simple systems

Ex. #1: Vehicle in motion



Figur: Force $u(t)$ and velocity $y(t)$.

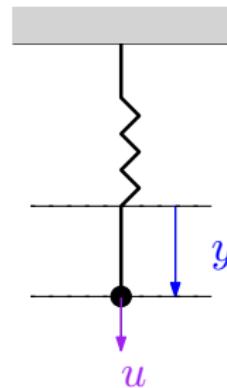
Standard form:

$$\frac{d}{dt}y + \left(\frac{C}{m}\right)y = \left(\frac{1}{m}\right)u$$

[Board: poles]

Build intuition from simple systems

Ex. #2: Damper



Figur: Force $u(t)$ and position $y(t)$.

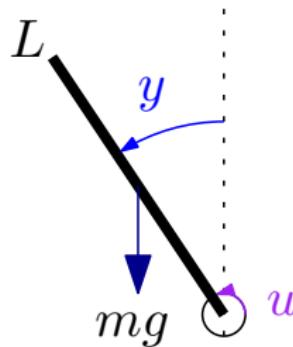
Standard form:

$$\frac{d^2}{dt^2}y + \left(\frac{K}{m}\right)y = \left(\frac{1}{m}\right)u$$

[Board: poles]

Build intuition from simple systems

Ex. #3: Inverted pendulum pendel



Figur: Torque $u(t)$ and angle $y(t)$.

Standard form (around $y \approx 0$):

$$\frac{d^2}{dt^2}y - \left(\frac{3g}{2L}\right)y = \left(\frac{3}{mL^2}\right)u$$

[Board: poles]

Summary and recap

- ▶ Transfer functions as a system description
- ▶ Poles and zeros
- ▶ (Bounded) input-output stability