



Intro. Computer Control Systems: F6

**Bode plot, design in frequency domain, Nyquist contour,
minimum phase**

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F5: Quiz!

- 1) General feedback control
 - a Leads to unstable closed-loop systems ↑
 - b Leads to more design freedom ↑
 - c Leads to non-minimum phase systems ↓

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- 1) General feedback control
 - a Leads to unstable closed-loop systems ↑
 - b Leads to more design freedom ↑
 - c Leads to non-minimum phase systems ↓
- 2) For linear time-invariant systems a sinusoidal input yields
 - a A sinusoidal output ↑
 - b A exponentially declining output ↑
 - c A stable output ↓

Properties in the frequency domain

- ▶ How does a (closed-loop) system respond to oscillating signals?

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$$r(t) = \frac{1}{2\pi} \int R(i\omega) e^{i\omega t} d\omega,$$

where

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \quad \text{and} \quad \omega = 2\pi f \quad (\text{frequency})$$

Properties in the frequency domain

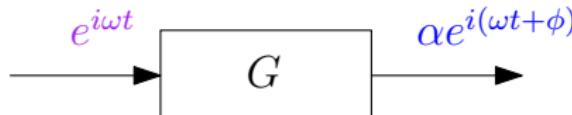
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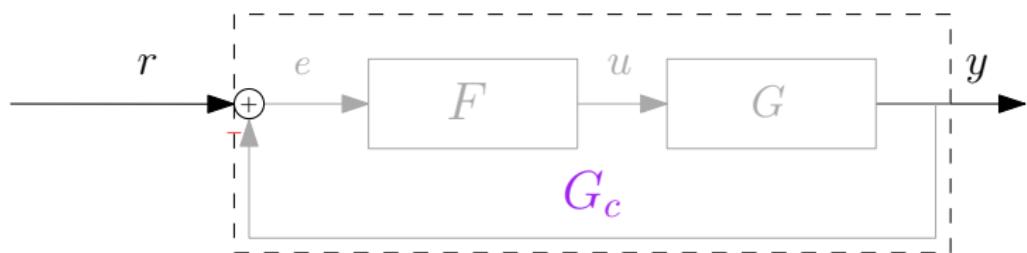
- ▶ $e^{i\omega t}$ is an eigen function to linear time-invariant systems:



- ▶ ⇒ System output $y(t)$ is reweighted sum of the *input cosine- and sine signals!*

Frequency response

Example of (closed-loop) system



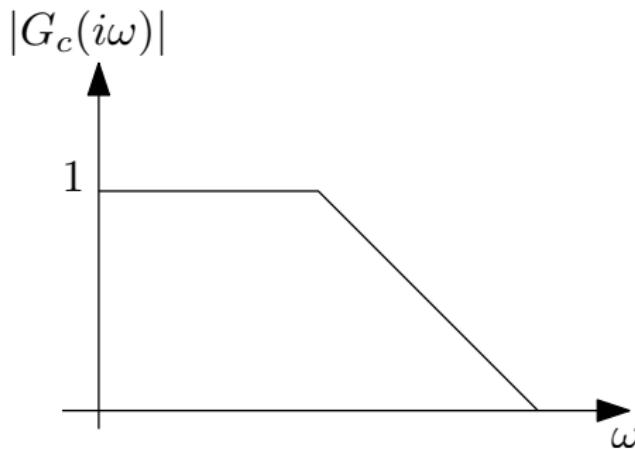
$$Y(s) = G_c(s)R(s)$$

Frequency response

Example of (closed-loop) system

$$\text{Frequency reponse: } G_c(s) \Big|_{s=i\omega} = G_c(i\omega)$$

Example: Magnitude curve of frequency response



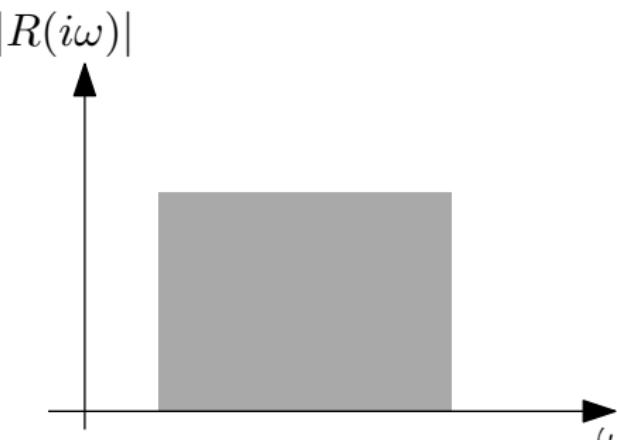
Frequency response

Example of (closed-loop) system

Signal as a weighted sum of cosine- and sine signals:

$$r(t) = \frac{1}{2\pi} \int R(i\omega) e^{i\omega t} d\omega.$$

Example: Frequency content of signal



Note that the plot is symmetric $|X(i\omega)| = |X(-i\omega)|$

Frequency response

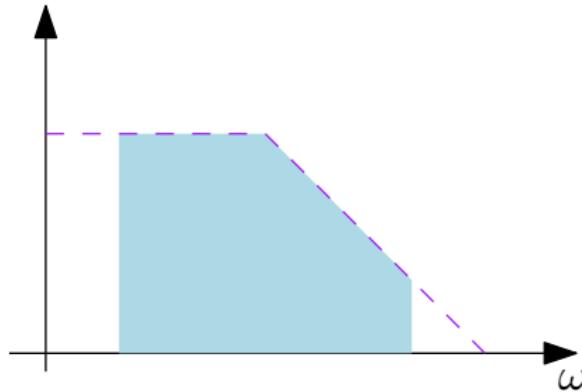
Example of (closed-loop) system

Signal as a weighted sum of cosine- and sine signals:

$$y(t) = \frac{1}{2\pi} \int Y(i\omega) e^{i\omega t} d\omega.$$

Example: Frequency content of output

$$|Y(i\omega)| = |G_c(i\omega)| |R(i\omega)|$$



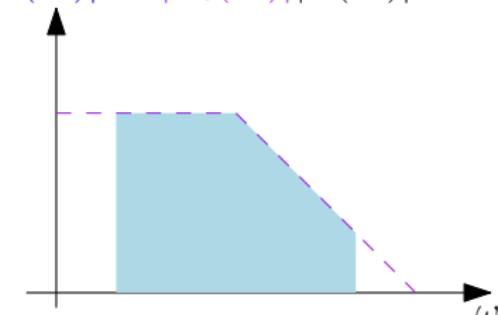
Frequency response

Example of (closed-loop) system

$$|R(i\omega)|$$



$$|Y(i\omega)| = |G_c(i\omega)| |R(i\omega)|$$

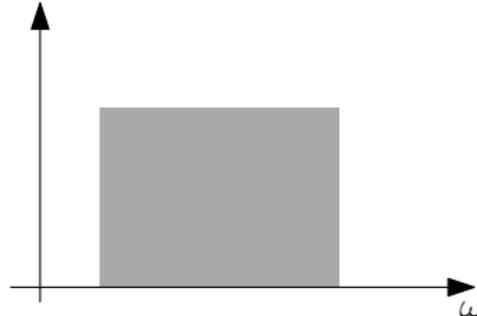


Cf. “sine-in sine-out”

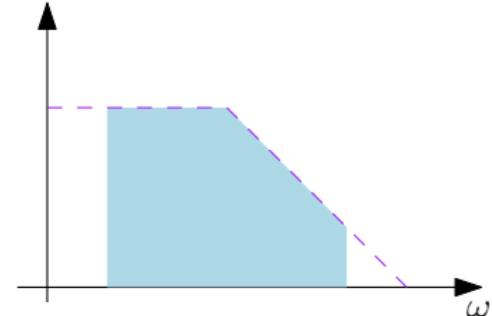
Frequency response

Example of (closed-loop) system

$$|R(i\omega)|$$



$$|Y(i\omega)| = |G_c(i\omega)| |R(i\omega)|$$



Recall goal of control: $y(t) \approx r(t) \Leftrightarrow Y(i\omega) \approx R(i\omega)$

- ▶ $Y(i\omega) = G_c(i\omega)R(i\omega)$
- ▶ Frequency response magnitude and phase
 $G_c(i\omega) = |G_c(i\omega)| e^{i\arg\{G_c(i\omega)\}}$
- ▶ Ideal: $|G_c(i\omega)| \approx 1$ and $\arg\{G_c(i\omega)\} \approx 0$

Poles/zeros and frequency response

Bode plot: recipe for magnitude curve

Re-write *standard form* using the poles and zeros:

$$\begin{aligned} G(s) &= \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = K_0 \frac{(s + z_1) \cdots (s + z_m)}{(s + p_1) \cdots (s + p_n)} \\ &= K \frac{(1 + \frac{s}{z_1}) \cdots (1 + \frac{s}{z_{m'}})}{s^q (1 + \frac{s}{p_1}) \cdots (1 + \frac{s}{p_{n'}})} \end{aligned}$$

Poles/zeros and frequency response

Bode plot: recipe for magnitude curve

System on *modified form*:

$$G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots}{s^q (1 + \frac{s}{p_1}) \cdots}$$

Logarithm of |frequency response|:

$$\begin{aligned} \log_{10} |G(i\omega)| &= \log_{10} |K| - q \log_{10} |\omega| \\ &\quad + \log_{10} \left| 1 + \frac{i\omega}{z_1} \right| + \cdots - \log_{10} \left| 1 + \frac{i\omega}{p_1} \right| - \cdots \end{aligned}$$

Intuition:

- ▶ When $\omega \ll |z_k|$ or $\omega \ll |p_k|$, the effect on $\log_{10} |G(i\omega)|$ is negligible
- ▶ When $\omega \gg |z_k|$ or $\omega \gg |p_k|$, increase/decrease $\log_{10} |G(i\omega)|$ with ω

Poles/zeros and frequency response

Bode plot: recipe for magnitude curve

Recipe for magnitude curve

$$\log_{10} |G(i\omega)| = \log_{10} |K| - q \log_{10} |\omega| + \log_{10} \left| 1 + \frac{i\omega}{z_1} \right| + \dots - \log_{10} \left| 1 + \frac{i\omega}{p_1} \right| - \dots$$

1. Sort $|z_k|$ or $|p_k|$ by distance to the origin.
2. Evaluate $\log |G(i\omega)|$ at the first $|z_1|$ or $|p_1|$ after the origin.
3. Plot the curve along $\omega \rightarrow \infty$:
 - ▶ For each zero $\omega \gg |z_k|$ the slope increases by $+1$.
 - ▶ For each pole $\omega \gg |p_k|$ the slope decreases by -1 .
 - ▶ Complex-conjugated poles give resonance peak at $\omega \approx |p_k|$ if $\zeta \ll 1$.

Bode plot

Example

1:st order system with simple poles.

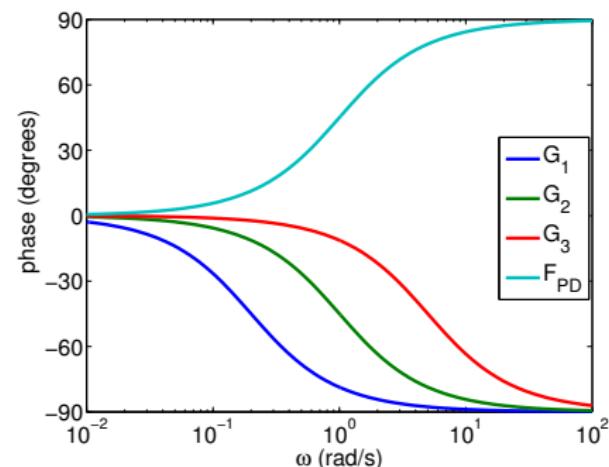
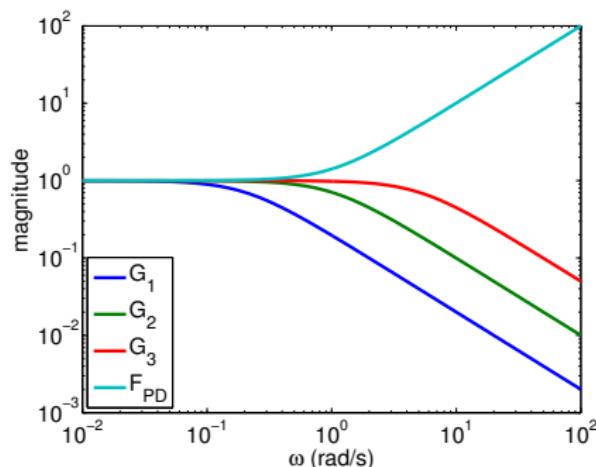
Examples:

$$G_1(s) = \frac{0.2}{s + 0.2},$$

$$G_2(s) = \frac{1}{s + 1},$$

$$G_3(s) = \frac{5}{s + 5},$$

$$(F_{PD}(s) = 1 + s)$$



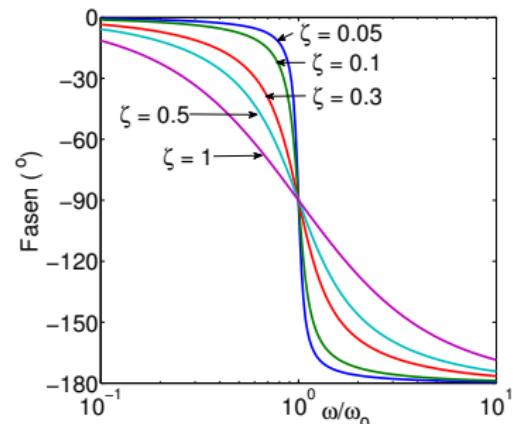
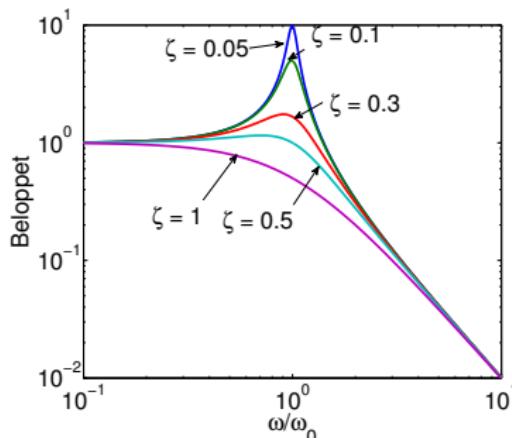
Bode plot

Example

2nd order system with complex conjugated poles:

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Poles $-\omega_0\zeta \pm i\omega_0\sqrt{1 - \zeta^2}$ där $|p_1| = |p_2| = \omega_0$.



$\zeta \ll 1 \Rightarrow$ gives resonance peak $\geq |G(i\omega_0)| = \frac{1}{2\zeta}$

Bode plot

Example

$$G(s) = \frac{100(s + 1)}{s(s^2 + 6s + 100)}$$

- ▶ Zeros: -1
- ▶ Poles:

$$0 \quad \text{and} \quad -3 \pm i\sqrt{91}$$

where $\omega_0 = 10$ and $\zeta = 0.3$.

[Board: sketch magnitude curve]

Bode plot

Example

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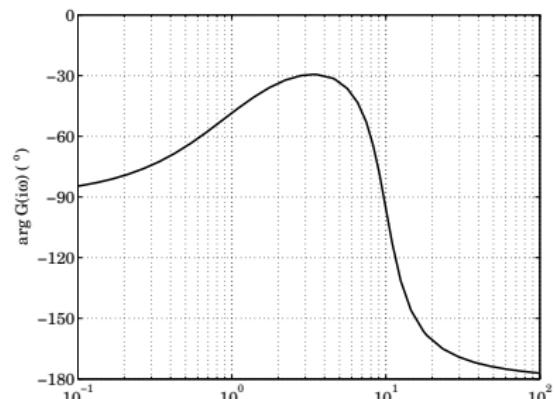
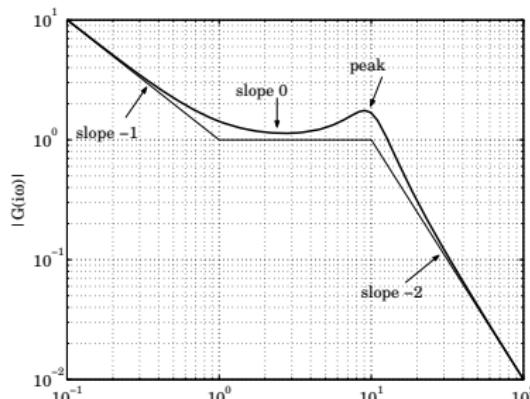
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where $\omega_0 = 10$ and $\zeta = 0.3$.

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Poles/zeros and frequency response

Bode plot: recipe for phase curve

System on *modified form*:

$$G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots}{s^q (1 + \frac{s}{p_1}) \cdots}$$

Argument of frequency response (in radians):

$$\arg\{G(i\omega)\} = -q \cdot \pi + \arctan \frac{\omega}{z_1} + \cdots - \arctan \frac{\omega}{p_1} - \cdots$$

Poles/zeros and frequency response

Bode plot: recipe for phase curve

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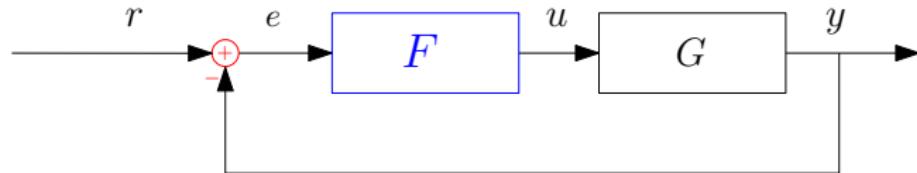
Plot phase curve If $G(s)$ does *not* have any poles/zeros in right half-plane:

- ▶ $\omega \rightarrow 0$: $\arg G(i\omega) \rightarrow 0$ if ing_s^1 and $G(0) > 0$
- ▶ $\omega \rightarrow \infty$: each zero or pole contributes with $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$ in phase, respectively.
- ▶ Each pole at the origin contributes with $-\frac{\pi}{2}$ in phase $\forall \omega$

Zeros in right half-plane gives negative phase contribution.

Design principles for controller

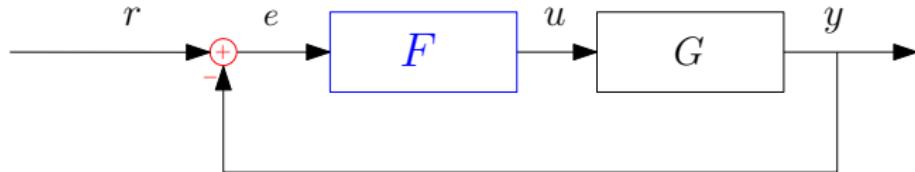
Feedback control system:



- ▶ Different performance metrics for G_c
- ▶ Different design principles of G_c via F

Design principles for controller

Feedback control system:



- ▶ Different performance metrics for G_c
- ▶ Different design principles of G_c via F
- ▶ *Ideal controller in the frequency domain:*

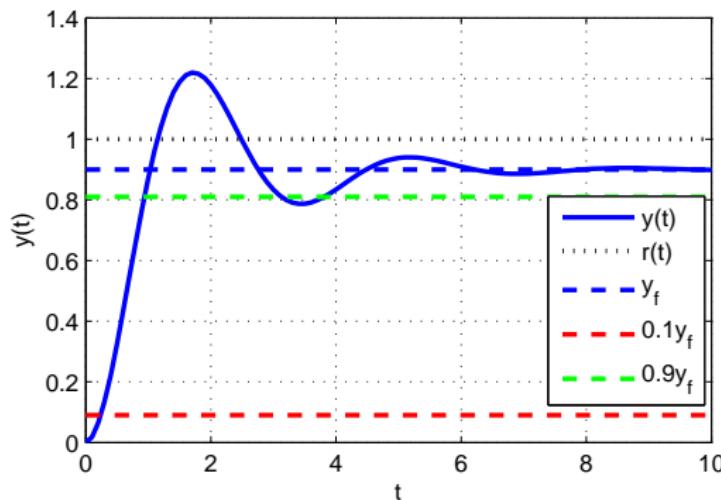
$$|G_c(i\omega)| \approx 1 \quad \text{and} \quad \arg\{G_c(i\omega)\} \approx 0$$

for the relevant frequencies ω .

Controller design in time domain

Specifications of the time response

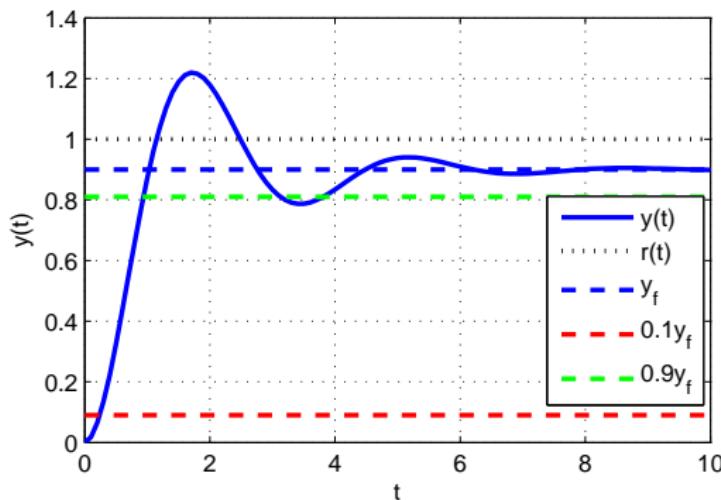
$y(t)$ when $r(t)$ is a step $\Leftrightarrow R(s) = \frac{r_0}{s}$:



Controller design in time domain

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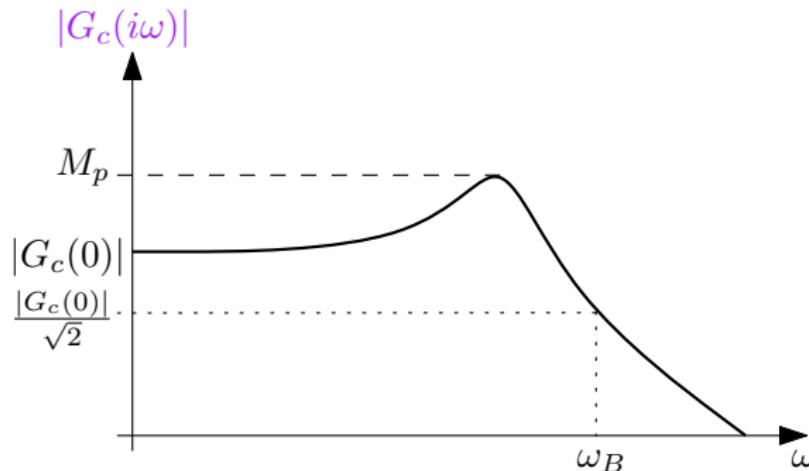
Performance metrics

- ▶ Quickness: **rise time** $T_r = t_{90\%} - t_{10\%}$
- ▶ Damping: **overshoot** $M = (y_{\max} - y_f)/y_f$
- ▶ Accuracy: **static control error** $e_f = r_0 - y_f$ (see F3!)

Controller design in frequency domain

Specifications of the frequency response

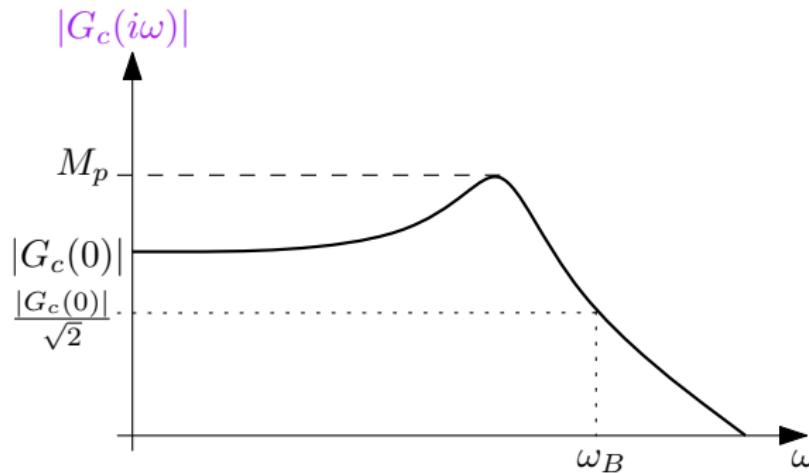
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Controller design in frequency domain

Specifications of the frequency response

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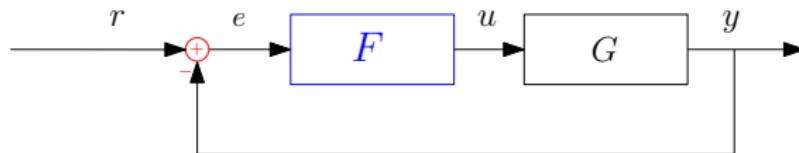
Performance metrics

- ▶ Quickness: **bandwidth** ω_B where $|G_c(i\omega_B)| = |G_c(0)|/\sqrt{2}$
- ▶ Damping: **resonance peak level** $M_p = \max(|G_c(i\omega)|)$
- ▶ Accuracy: **static gain** $G_c(0)$ (see F3!)

Design G_c via open-loop system G_o

Frequency response and Nyquist contour

Feedback control system:

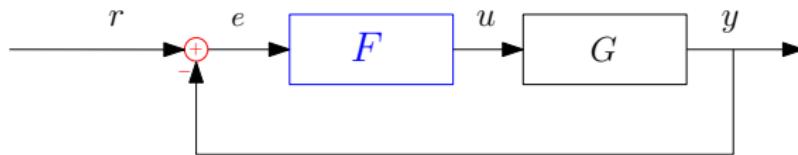


- ▶ Closed-loop: $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$
- ▶ Open-loop: $G_o(s) = F(s)G(s)$

Design G_c via open-loop system G_o

Frequency response and Nyquist contour

Feedback control system:



- ▶ Closed-loop: $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$
- ▶ Open-loop: $G_o(s) = F(s)G(s)$
- ▶ $G_c(s)$ stable $\iff 1 + G_o(s)$ no roots in right half-plane.

Definition: Nyquist contour

Complex-valued frequency response, $G_o(i\omega)$, as a function of $0 \leq \omega < \infty$.

Nyquist contour

Example

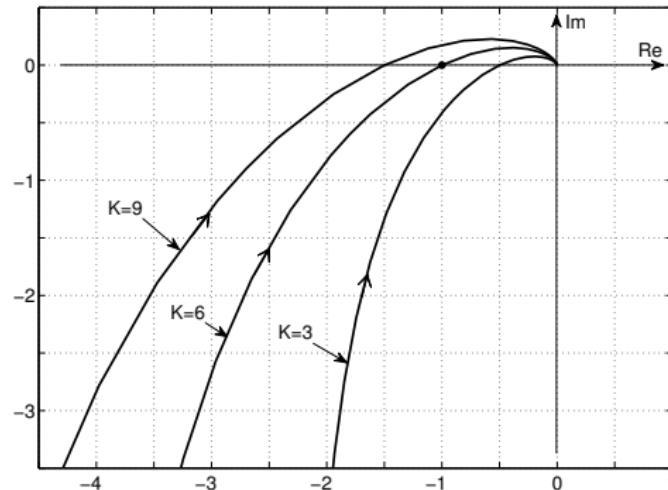
DC-motor $G(s) = \frac{1}{s(s+1)}$ with controller $F(s) = \frac{K}{s+2}$.

Nyquist contour

Example

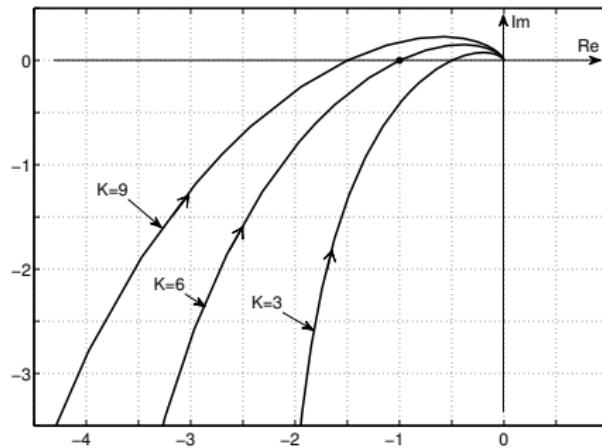
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Nyquist contour for $G_o(i\omega) = G(i\omega)F(i\omega)$:



Closed-loop system: $K = 3$ (stable), $K = 9$ (unstable), and $K = 6$ (marginally stable).

Nyquist criterion

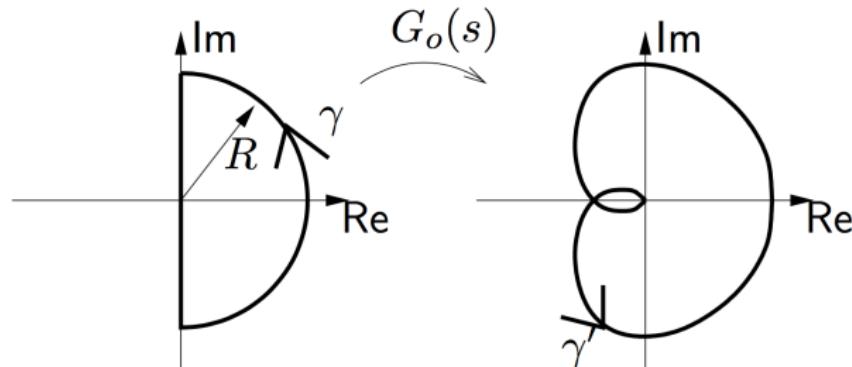


(Result 3.3) Basic Nyquist criterion:

If $G_o(s)$ has no poles in right half-plane: $G_c(s)$ stable \Leftrightarrow Nyquist contour $G_o(i\omega)$ does not encircle -1

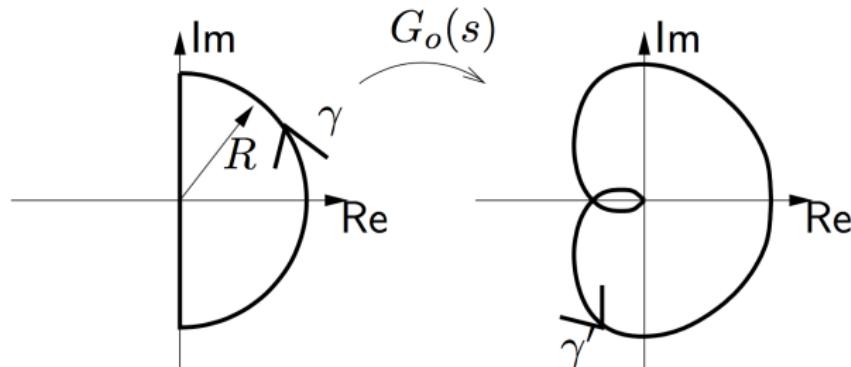
Nyquist criterion

In general: Let s form semi-circle γ , with radius $R \rightarrow \infty$. Then Nyquist contour $G_o(s)$ is γ' .



Nyquist criterion

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(Result 3.3) General Nyquist criterion:

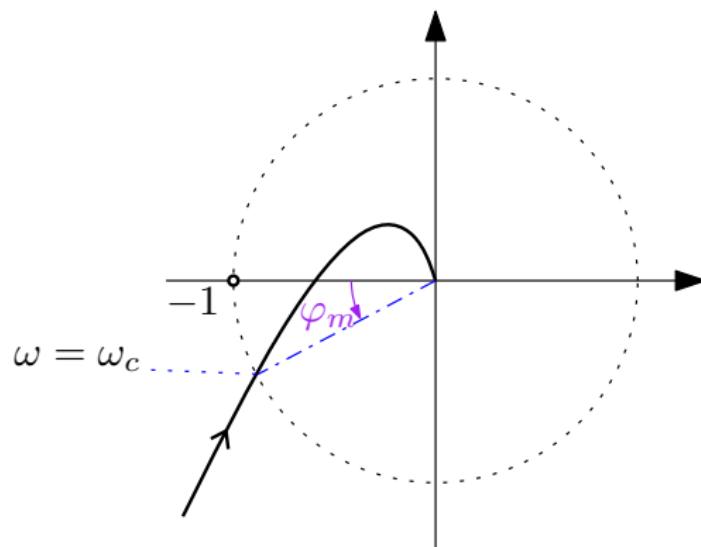
#poles{ $G_c(s)$ } in right half-plane = #poles{ $G_o(s)$ } in right half-plane + Number of positive circles of γ' around -1

Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

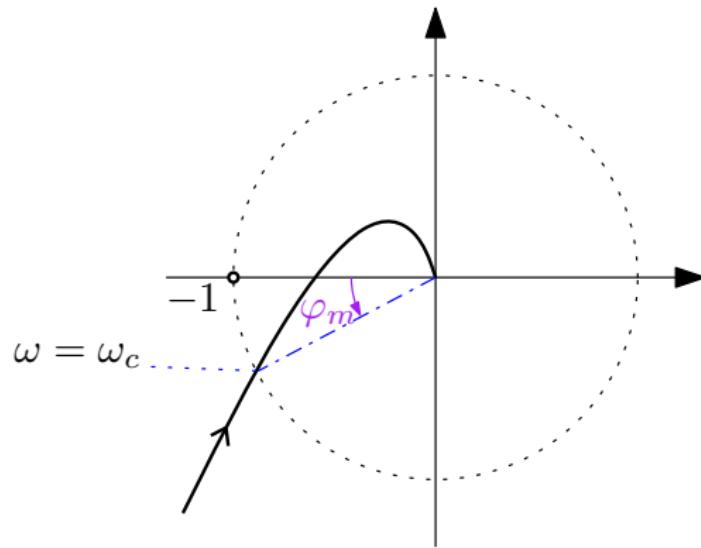
Distance to -1 affect resonance peak:

$$|G_c(i\omega)| = \frac{|G_o(i\omega)|}{|1 + G_o(i\omega)|} = \frac{|G_o(i\omega)|}{\underbrace{|G_o(i\omega) - (-1)|}_{\text{distance to } -1}}$$



Controller design via Nyquist/Bode plot

Specification via $G_o(s)$



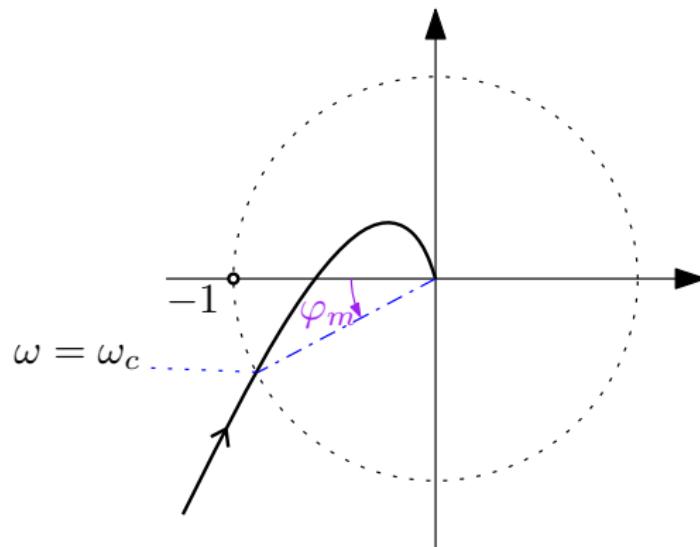
At $\omega = \omega_c$, we have magnitude and phase:

- ▶ $|G_o(i\omega_c)| = 1$
- ▶ $\arg\{G_o(i\omega_c)\} = -180^\circ + \varphi_m$

Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

Nyquist contour $G_o(i\omega)$:



Open-loop design $G_o(s) = F(s)G(s) \Rightarrow$ closed-loop properties:

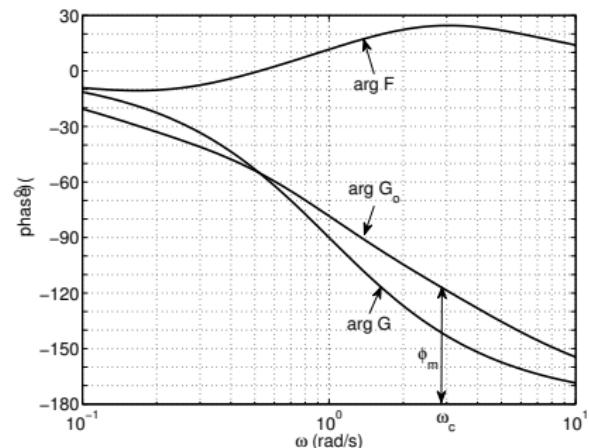
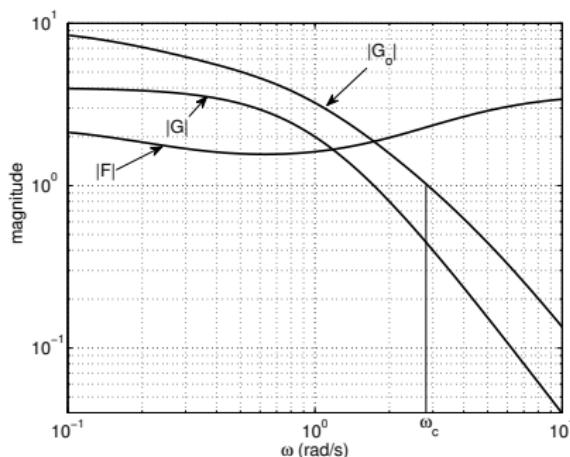
- ▶ Crossover frequency $\omega_c \Rightarrow$ bandwidth ω_B (quickness)
- ▶ Phase margin $\varphi_m \Rightarrow$ resonance peak M_p (damping)

Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

Open-loop frequency characteristics that we can shape:

- ▶ Crossover frequency ω_c
- ▶ Phase margin φ_m



Note $\log_{10} |G_o| = \log_{10} |G| + \log_{10} |F|$ and
 $\arg\{G_o\} = \arg\{G\} + \arg\{F\}$

Design in the frequency domain

Lead-lag controller

Controller structure:

$$F(s) = \underbrace{K}_{\text{P-control}} \quad F_{\text{lead}}(s) F_{\text{lag}}(s)$$

where

- ▶ $F_{\text{lead}}(s)$ adjusts φ_m (damping) and ω_c (quickness)
- ▶ $F_{\text{lag}}(s)$ adjusts $G_o(0)$ (accuracy)

Note that

$$\begin{aligned} |G_o(i\omega)| &= |G(i\omega)| K |F_{\text{lead}}(i\omega)| |F_{\text{lag}}(i\omega)| \\ \arg\{G_o(i\omega)\} &= \arg\{G(i\omega)\} + \arg\{F_{\text{lead}}(i\omega)\} + \arg\{F_{\text{lag}}(i\omega)\}, \end{aligned}$$

shape the Bode plot/Nyquist contour of G_o

Design in the frequency domain

Lead-lag controller

Controller structure:

$$F(s) = \underbrace{K}_{\text{P-control}} \quad \color{blue}{F_{\text{lead}}(s)} \color{red}{F_{\text{lag}}(s)}$$

► Lead filter

$$\color{blue}{F_{\text{lead}}(s)} = \frac{\tau_D s + 1}{\beta \tau_D s + 1}, \quad 0 \leq \beta < 1, \quad \tau_D > 0$$

increases phase margin φ_m and crossover ω_c .

► Lag filter

$$\color{red}{F_{\text{lag}}(s)} = \frac{\tau_I s + 1}{\tau_I s + \gamma}, \quad 0 \leq \gamma < 1, \quad \tau_I > 0$$

increases static gain.

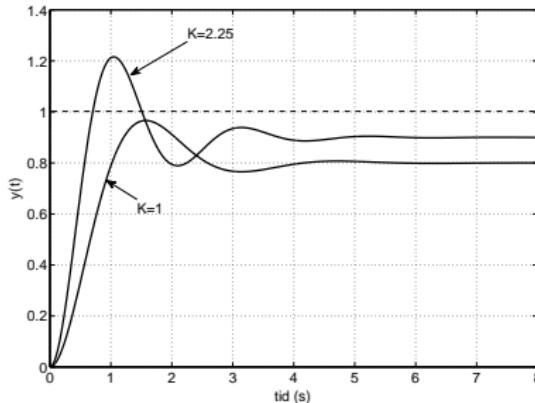
► See ch. 5.4 for tuning principles

Lead-lag design using Bode plot

Example of lead-lag controller

System $Y(s) = \frac{4}{(s+1)^2} U(s)$ with controller

$U(s) = F(s)(R(s) - Y(s))$ where $F(s) = K$.



Performance specifications when $r(t)$ is a step:

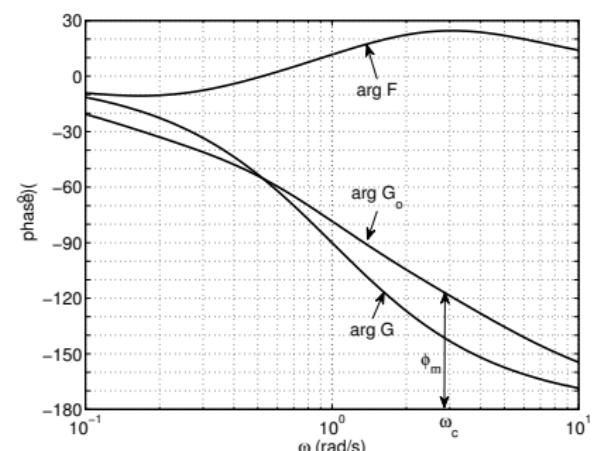
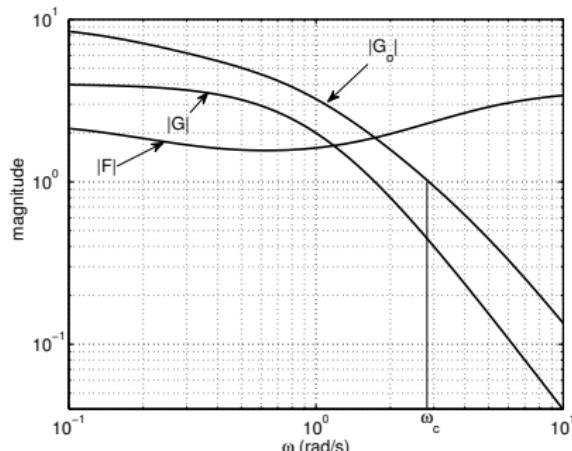
- ▶ Accuracy, static error $e_f \leq 0.1$,
- ▶ Quickness, rise time $T_r \leq 0.5$,
- ▶ Damping, overshoot $M \leq 20\%$.

Lead-lag design using Bode plot

Example of lead-lag controller

Bode plots for

- ▶ system $G(s)$,
- ▶ open-loop $G_o(s) = F(s)G(s)$,
- ▶ build lead-lag controller $F(s)$ that compensates P -part

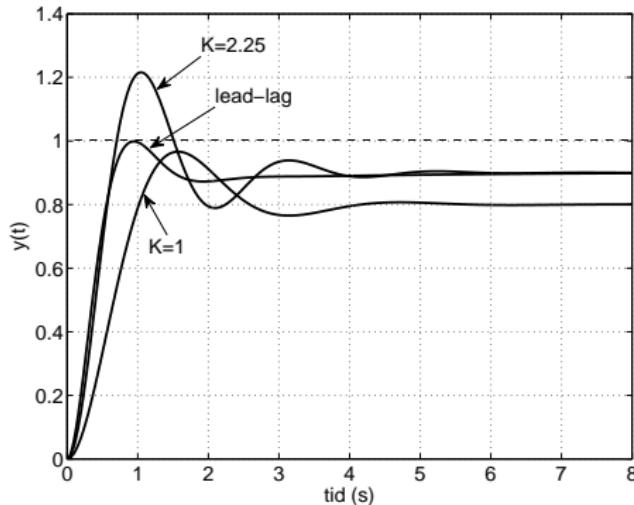


Lead-lag design using Bode plot

Example of lead-lag controller

Step response $y(t)$ with lead-lag controller

$$F(s) = K F_{\text{lead}}(s) F_{\text{lag}}(s).$$



Performance specifications are met with lead-lag controller:

- ▶ $T_r = 0.45$ seconds, $M = 8\%$, $e_f < 0.1$.



Minimum phase systems

Several different systems $G(s)$ may have identical magnitude curve $|G(i\omega)|$. Only phase curve $\arg\{G(i\omega)\}$ will differ.

Definition:

Among all systems $G(s)$ with the *same magnitude curve*, the system with the *least negative phase shift* is a **minimum phase system**.



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Cf. bicycle as non-minimum phase!

Non-minimum phase systems

Example

Exemple of open-loop systems

- ▶ $G_o(s) = \frac{s+1}{s(s^2+2s+2)}$: $p_i = 0, -1 \pm i$. $z_i = -1$
- ▶ $G'_o(s) = \frac{-s+1}{s+1} \cdot G_o(s)$: $p_i = 0, -1 \pm i$. $z_i = +1$
- ▶ $G''_o(s) = e^{-2s}G_o(s)$: as above but time-delay 2 sec.

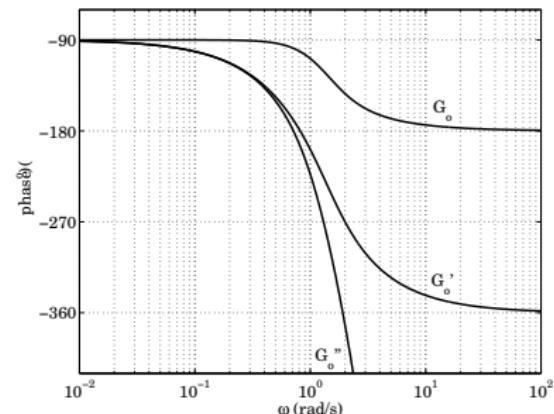
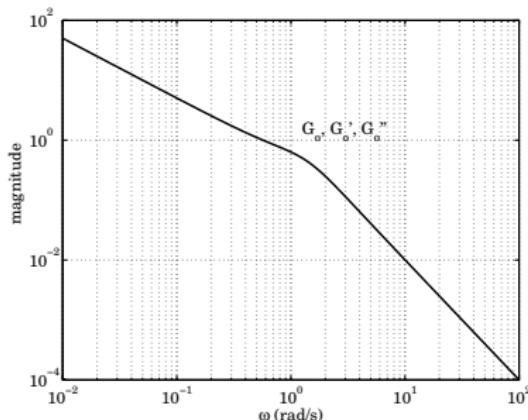
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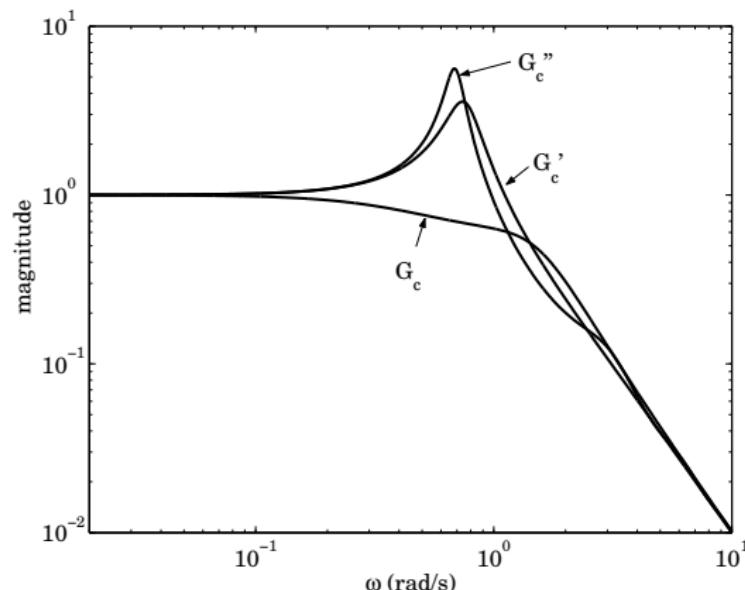
Bode plots for G_o , G'_o & G''_o :



Non-minimum phase systems

Examples

Magnitude curves for corresponding closed-loop systems, G_c , G'_c & G''_c :



Non-minimum phase systems are hard to control!

Summary and recap

- ▶ Frequency response and Bode plots
- ▶ Performance metrics in the frequency domain
 - ▶ Bandwidth
 - ▶ Resonance peak
 - ▶ Static gain
- ▶ Open-loop design and Nyquist contour
- ▶ Minimum phase systems