



Intro. Computer Control Systems: F8

Properties of state-space descriptions and feedback

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F7: Quiz!

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- 1) The state-space description of a system is
- a not unique ↑
 - b unique ↑
 - c stable ↓

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 - a poles ↑
 - b zeros ↑
 - c the closed-loop system ↓



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- 1) The state-space description of a system is
 - a not unique ↑
 - b unique ↑
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 - 2) The eigenvalues of the system matrix A reveals something about
 - a poles ↑
 - b zeros ↑
 - c the closed-loop system ↓
 - 3) Solution to $\dot{x} = Ax + Bu$ with initial condition x_0 is obtained using
 - a a linear system of equations ↑
 - b the matrix exponential ↑
 - c the Nyquist contour ↓

Nonlinear systems and states

Most systems are nonlinear!



Nonlinear differential equations:

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

Linearize around *operating point* x_0, u_0 . Typically use a **stationary point**: $\dot{x} = f(x_0, u_0) = 0$

Nonlinear systems and states

Nonlinear differential equations:

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u)\end{aligned}$$

Taylor series expansion around *stationary point* x_0, u_0 with $y_0 = h(x_0, u_0)$ results in **linear deviation model**:

$$\boxed{\begin{aligned}\dot{\Delta x} &= A\Delta x + B\Delta u \\ \Delta y &= C\Delta x + D\Delta u\end{aligned}}$$

- ▶ Linear state-space description of the deviations **around** the operating point of system.
- ▶ Matrices A, B, C and D given by **derivatives** of $f(x, u)$ and $h(x, u)$ with respect to x and u . **See ch. 8.4 G&L.**

State-feedback control

State space description of linear time-invariant system

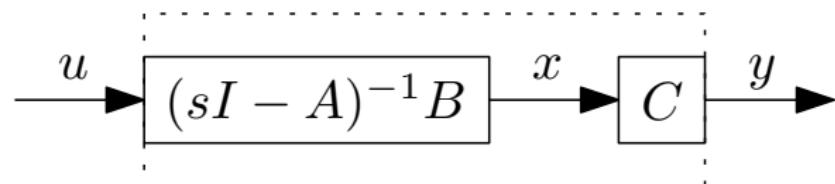
$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\Rightarrow Y(s) = G(s)U(s)$$



State-feedback control

State space description of linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\quad \Rightarrow \quad G(s) = C(sI - A)^{-1}B$$

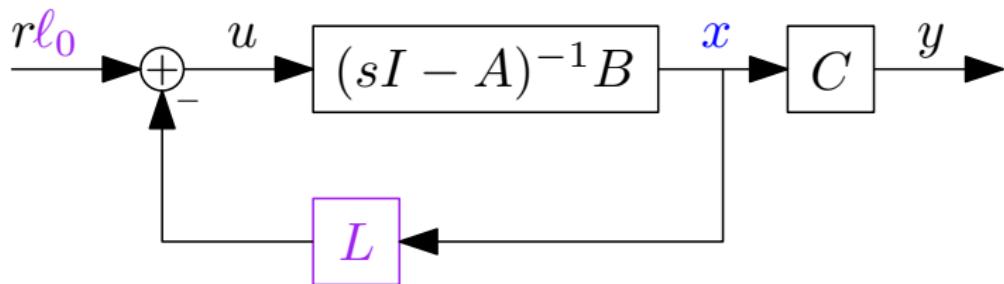


State-feedback control

Idea: Feedback control using states

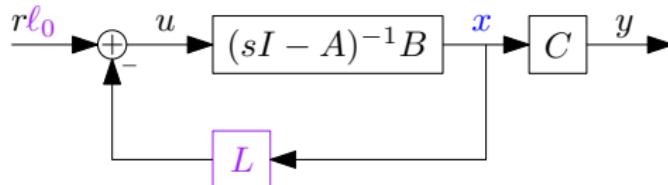
$$u = -Lx + l_0 r,$$

where L and l_0 are design parameters.



$$\dot{x} = Ax + B \underbrace{(-Lx + l_0 r)}_{=u}$$

State-feedback control



Closed-loop system from r to y comes:

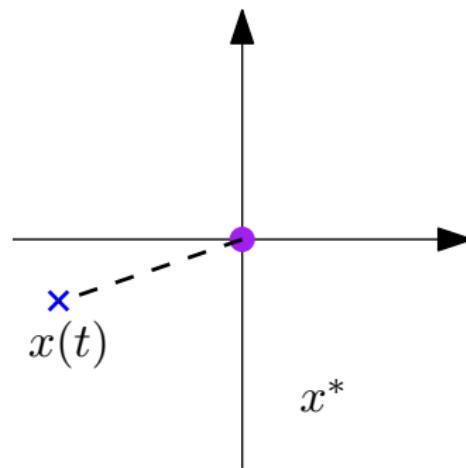
$$\begin{aligned}\dot{x} &= Ax + B(-Lx + l_0r) = (A - BL)x + Bl_0r \\ y &= Cx\end{aligned}$$

Is it possible to

- ▶ control the system to all states x^* in \mathbb{R}^n ?
- ▶ design the closed-loop system's poles?
- ▶ (estimate the state $x(t)$?)

Controllability

A sought state x^* is **controllable** if some input $u(t)$ can move the system from $x(0) = 0$ to $x(T) = x^*$



Controllability

For $x_0 = 0$, we can compute the state at $t = T$

$$x(T) = e^{At}x_0 + \int_0^T e^{A\tau}Bu(T-\tau)d\tau$$



Controllability

Med $x_0 = 0$ är tillståndet vid $t = T$

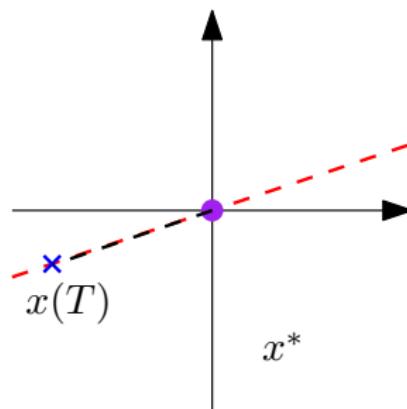
$$\begin{aligned}x(T) &= \int_0^T e^{A\tau} Bu(T - \tau) d\tau \\&= [\text{via Cayley-Hamiltons theorem}] \\&= B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}\end{aligned}$$

Therefore:

- ▶ $x(T)$ is a linear combination of $B, AB, \dots, A^{n-1}B$.
- ▶ A state x^* is controllable if it can be expressed as such a linear combination, i.e., if x^* is in the column space of

$$\boxed{\mathcal{S} \triangleq [B \ AB \ \cdots \ A^{n-1}B]}$$

Controllability



Figur: Example column space of \mathcal{S} and non-controllable state x^* .

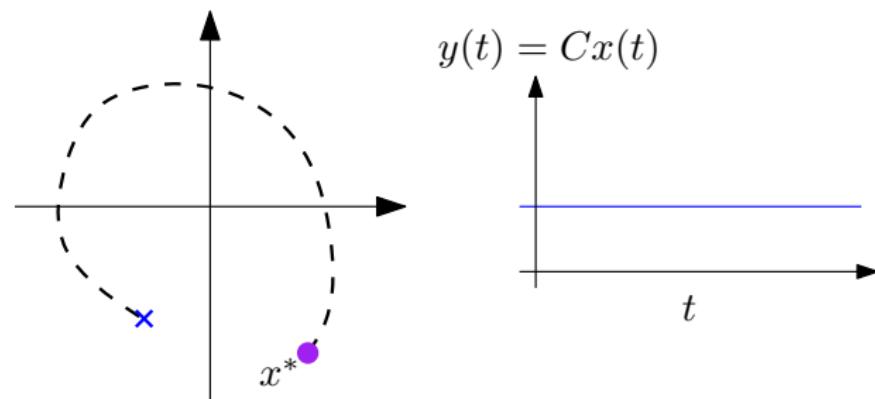
Controllable system

All states x^* are controllable $\Leftrightarrow \mathcal{S}$:s columns are linearly independent

Note: $\text{rank}(\mathcal{S}) = n$ or $\det(\mathcal{S}) \neq 0$

Observability

Assume $u(t) \equiv 0$. A state $x^* \neq 0$ is **unobservable** if the output $y(t) \equiv 0$ when system starts at $x(0) = x^*$.



Observability

When $u(t) \equiv 0$ we obtain

$$\begin{aligned}y(t) &= Cx(t) \\&= Ce^{At}x^* + 0\end{aligned}$$

When $y(t) \equiv 0$ we do not observe any changes in the output:

$$\frac{d^k}{dt^k}y(t)\Big|_{t=0} = CA^kx^* = 0.$$

That is,

$$Cx^* = 0, \quad CAx^* = 0, \quad \dots, \quad CA^{n-1}x^* = 0$$



Observability

When $u(t) \equiv 0$ and $y(t) \equiv 0$ we observe no changes:

$$Cx^* = 0, \quad CAx^* = 0, \quad \dots, \quad CA^{n-1}x^* = 0$$

or

$$\mathcal{O}x^* = 0$$

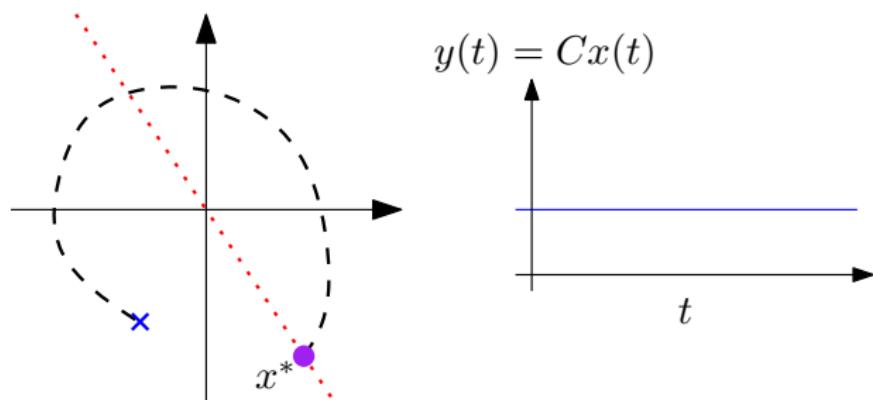
where

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Therefore:

- ▶ A state $x^* \neq 0$ is **unobservable** if it belongs to the **null space** of \mathcal{O} .

Observability



Figur: Example null space of \mathcal{O} and unobservable state x^* .

Observable system

All states x^* are observable $\Leftrightarrow \mathcal{O}$:s columns are linearly independent

Note: $\text{rank}(\mathcal{O}) = n$ or $\det(\mathcal{O}) \neq 0$

Build intuition from simple systems

Exemple: controllable system

System on controllable canonical form \Leftrightarrow controllable

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= [1 \quad 1] x(t)\end{aligned}$$

Transfer function:

$$G(s) = C(sI - A)^{-1}B = \frac{s + 1}{s^2 + 2s + 1} = \frac{s + 1}{(s + 1)^2} = \frac{1}{s + 1}$$

[Board: investigate observability using \mathcal{O}]

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[Board: investigate observability using \mathcal{O}]

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \det \mathcal{O} = 0 \Leftrightarrow \text{unobservable}$$

Build intuition from simple systems

Example: observable system

System on observable canonical form \Leftrightarrow observable

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [1 \quad 0] x(t)\end{aligned}$$

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[Board: investigate controllability using \mathcal{S}]

$$\mathcal{S} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \det \mathcal{S} = 0 \Leftrightarrow \text{non-controllable}$$

Build intuition from simple systems

Exemple: controllable and observable system

Systems in previous examples have the same transfer function

$$G(s) = \frac{1}{s + 1}.$$

Can also be written in state-space form

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t), \\ y(t) &= x(t).\end{aligned}$$

where $x(t)$ is a scalar.

[Board: investigate \mathcal{S} and \mathcal{O}]

Build intuition from simple systems

Exemple: controllable and observable system

Systems in previous examples have the same transfer function

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[Board: investigate \mathcal{S} and \mathcal{O}]

$$\begin{array}{lcl} \mathcal{S} = 1 & \Rightarrow & \det \mathcal{S} = 1 \\ \mathcal{O} = 1 & \Rightarrow & \det \mathcal{O} = 1 \end{array} \Leftrightarrow \text{controllable and observable} \quad (1)$$

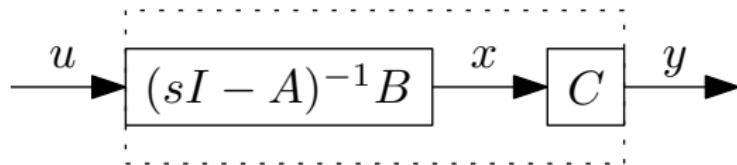
Note: we eliminated “invisible states”

Minimal realization

System with transfer function $G(s)$ and state-space form

$$\dot{x} = Ax + Bu$$

$$y = Cx$$



Definition 8.2 G&L

State-space form of $G(s)$ is a **minimal realization** if vector x has the smallest possible dimension.

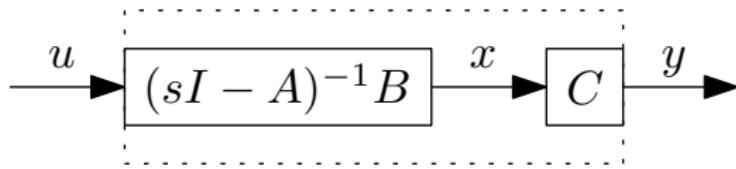


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Definition 8.2 G&L

State-space form of $G(s)$ is a **minimal realization** if vector x has the smallest possible dimension.

Result 8.11(+8.12) G&L

A state-space form is **minimal realization** \Leftrightarrow controllable and observable $\Leftrightarrow A$:s eigenvalues = $G(s)$:s poles

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \ell_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

gives **closed-loop system**

$$\dot{x} = (A - B\textcolor{blue}{L})x + B\ell_0 r$$

$$y = Cx$$

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \ell_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

gives **closed-loop system**

$$\dot{x} = (A - B\textcolor{blue}{L})x + B\ell_0 r$$

$$y = Cx$$

That is, $Y(s) = \textcolor{violet}{G}_c(s)R(s)$ where

$$\textcolor{violet}{G}_c(s) = C(sI - A + B\textcolor{blue}{L})^{-1}B\ell_0$$

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \ell_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

gives **closed-loop system**

$$\begin{aligned}\dot{x} &= (A - B\textcolor{blue}{L})x + B\ell_0 r \\ y &= Cx\end{aligned}$$

That is, $Y(s) = \textcolor{pink}{G}_c(s)R(s)$ where

$$\textcolor{pink}{G}_c(s) = C(sI - A + B\textcolor{blue}{L})^{-1}B\ell_0$$

Eigenvalues/**poles** given by polynomial equation

$$\boxed{\det(sI - A + B\textcolor{blue}{L}) = 0}$$

which we can *design* via $\textcolor{blue}{L}$!

State-feedback control

Design of the gain ℓ_0

- ▶ $Y(s) = \textcolor{red}{G}_c(s)R(s)$ where

$$\textcolor{red}{G}_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

- ▶ It is *desirable* to have at least $\textcolor{red}{G}_c(0) = 1$

State-feedback control

Design of the gain ℓ_0

- ▶ $Y(s) = G_c(s)R(s)$ where

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

- ▶ It is *desirable* to have at least $G_c(0) = 1$
- ▶ $G_c(0) = C(-A + BL)^{-1}B\ell_0 = 1$ and so

$$\boxed{\ell_0 = \frac{1}{C(-A + BL)^{-1}B}}$$

State-feedback control

Design of the gain ℓ_0

- $Y(s) = G_c(s)R(s)$ where

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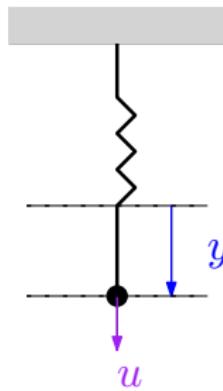
- It is *desirable* to have at least $G_c(0) = 1$
- $G_c(0) = C(-A + BL)^{-1}B\ell_0 = 1$ and so

$$\boxed{\ell_0 = \frac{1}{C(-A + BL)^{-1}B}}$$

- More generally, replace $\ell_0 r$ with $F_r(s)R(s)$
- How to design L ?

Build intuition from simple systems

Exemple: state-vector in \mathbb{R}^2



Figur: Force $u(t)$ and position $y(t)$.

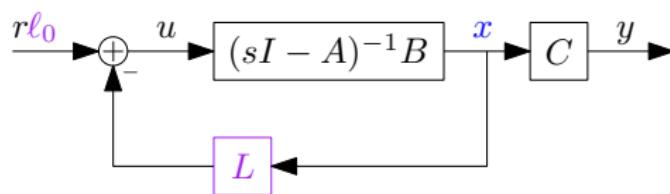
State-space form:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\end{aligned}$$

[Board: design L so that closed-loop system has poles -2 and -3]

Pole placement

State-feedback control

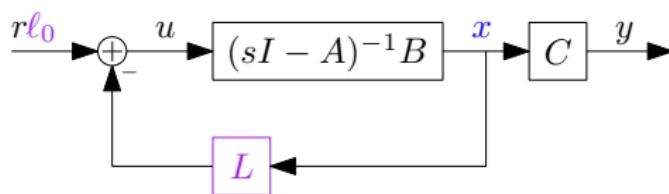


Result 9.1

State-space form is **controllable** $\Leftrightarrow L$ can be designed to yield **arbitrarily placed poles** (real and complex-conjugated) of the closed-loop system

Pole placement

State-feedback control



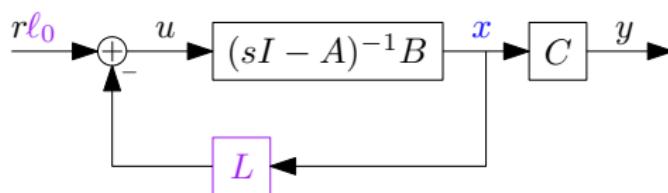
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State-space form is **controllable** $\Leftrightarrow L$ can be designed to yield **arbitrarily placed poles** (real and complex-conjugated) of the closed-loop system

- ▶ L solved by $\det(sI - A + BL) = 0$ with **desired roots**
- ▶ L **very simple** to solve for system on controllable canonical form

Pole placement

State-feedback control



Result 9.1

State-space form is **controllable** $\Leftrightarrow L$ can be designed to yield **arbitrarily placed poles** (real and complex-conjugated) of the closed-loop system

- ▶ L solved by $\det(sI - A + BL) = 0$ with **desired roots**
- ▶ L **very simple** to solve for system on controllable canonical form
- ▶ What to do when we **can't** measure x directly?

Summary and recap

- ▶ Linearization of nonlinear system models
- ▶ Properties:
 - ▶ Controllable
 - ▶ Observable
 - ▶ Minimal realization
- ▶ State-feedback control
- ▶ Pole placement for the closed-loop system