



# Intro. Computer Control Systems: F9

**State-feedback control and observers**

Dave Zachariah

Dept. Information Technology, Div. Systems and Control

# F8: Quiz!

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1) For an **observable** system

- a the effect of all  $x(t)$  can be observed in  $y(t)$  ↑
- b we have  $\det \mathcal{O} = 0$  ↑
- c we have stability ↓



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  - c we have stability ↓
  
- 2) If a state-space form of  $G(s)$  is a **minimal realization**,
  - a  $A$ :s eigenvalues <  $G(s)$ :s poles ↑
  - b  $A$ :s eigenvalues =  $G(s)$ :s poles ↑
  - c there exists more compact state-space forms ↓



# F8: Quiz!

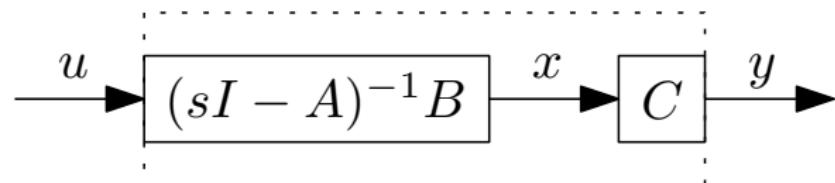
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  - b  $A$ :s eigenvalues =  $G(s)$ :s poles ↑
  - c there exists more compact state-space forms ↓
- 3) For a **controllable** system with state-feedback control
  - a no information about the system is required ↑
  - b the poles of the closed-loop system can be designed arbitrarily ↑
  - c the closed-loop system is stable ↓

# State-feedback control

State-space form of linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\Rightarrow G(s) = C(sI - A)^{-1}B$$



# State-feedback control

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Controler using state feedback

$$u = -\textcolor{magenta}{L}x + \ell_0 r$$

gives **closed-loop system**

$$\begin{aligned}\dot{x} &= (A - B\textcolor{magenta}{L})x + B\ell_0 r \\ y &= Cx\end{aligned}$$

where  $r$  is the reference signal.

# State-feedback control

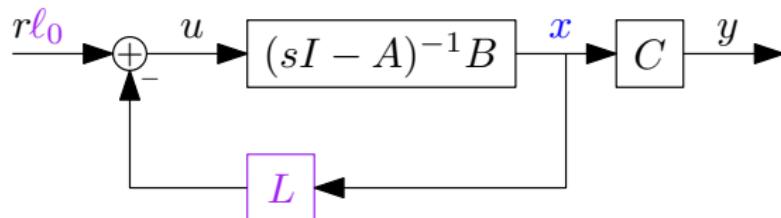
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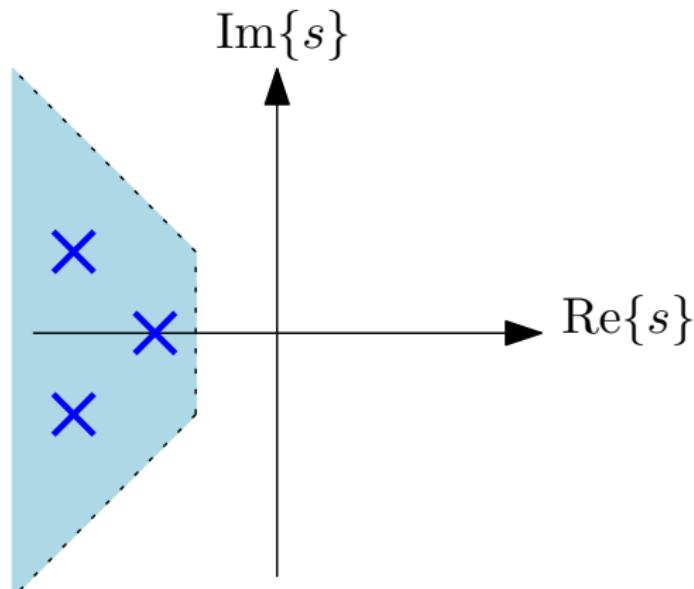


$$G_c(s) = C(sI - A + B\textcolor{magenta}{L})^{-1}B\ell_0$$

# Pole placement

## Rules of thumb for designing $L$

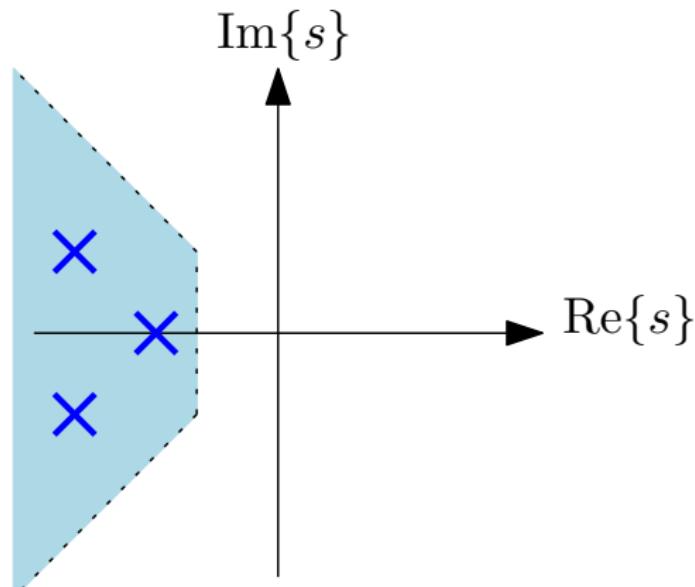
Eigenvalues/poles given by  $\det(sI - A + BL) = 0$ , which we can design



# Pole placement

## Rules of thumb for designing $L$

Eigenvalues/**poles** given by  $\det(sI - A + BL) = 0$ , which we can *design*



Distance to the origin: Quick system but also sensitive to disturbances

# Estimating the states

## Via simulation

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- ▶ Controller

$$u = -Lx + \ell_0 r$$

requires states  $x$  which are often **unknown**.

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where  $\hat{x}$  is an **estimate** of  $x$ .

# Estimating the states

## Via simulation

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where  $\hat{x}$  is an **estimate** of  $x$ .

- ▶ *Naive idea:* Estimate  $x$  by *simulating* the states

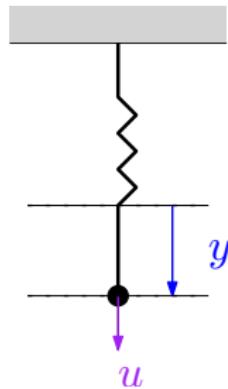
$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \hat{x}(0) = \hat{x}_0$$

where  $\hat{x}_0$  is an **initial guess**.

# Build intuition from simple systems

## State estimation via simulation

Ex.: Damper



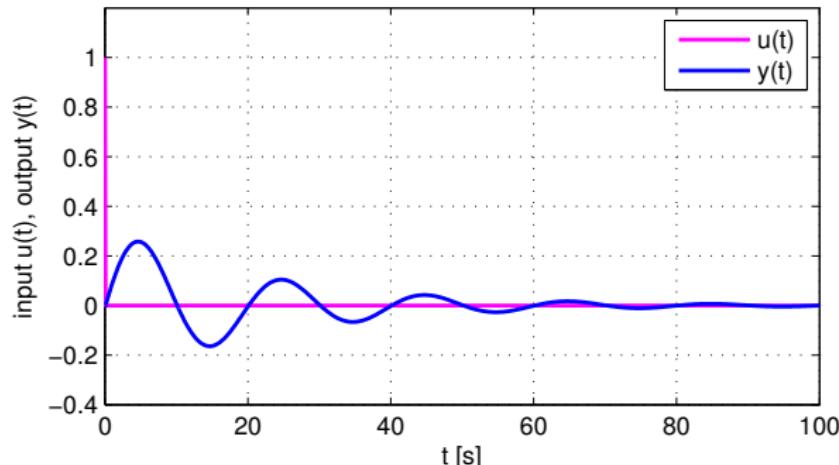
State-space form:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t), \quad x(0) = x_0 \\ y(t) &= [1 \quad 0] x(t)\end{aligned}$$

# Build intuition from simple systems

## State estimation via simulation

Example using impulse  $u$



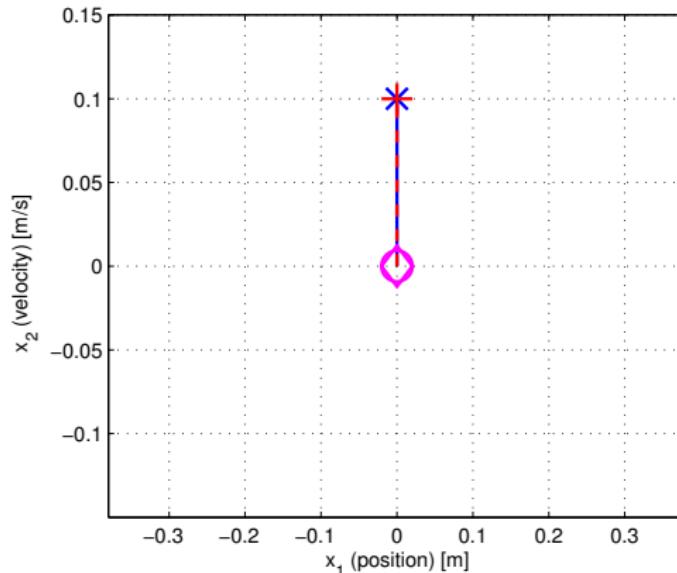
System with unknown initial state  $x_0$

# Build intuition from simple systems

## State estimation via simulation

Naive estimate using *perfect* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \boxed{\hat{x}_0 = x_0}$$



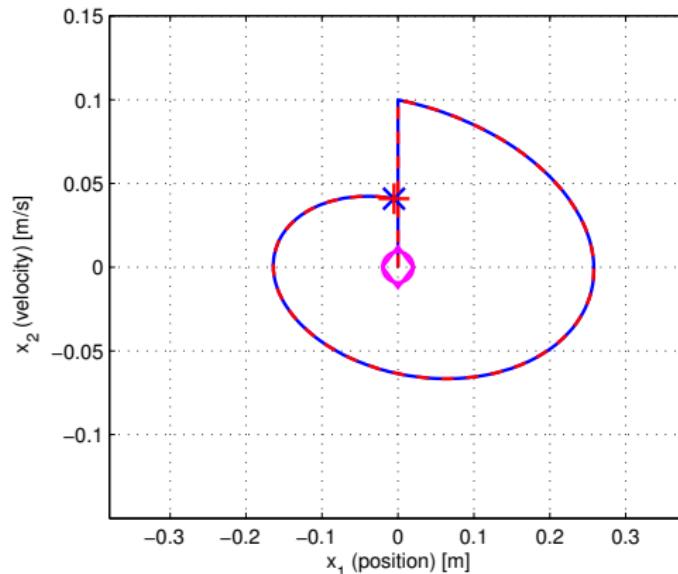
$x$  versus  $\hat{x}$  at  $t = 0^+$

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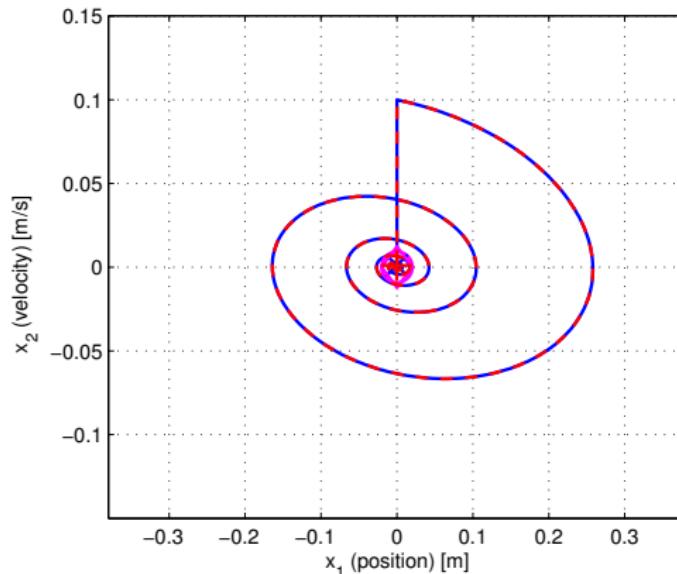
$x$  versus  $\hat{x}$  at  $t = 20$

# Build intuition from simple systems

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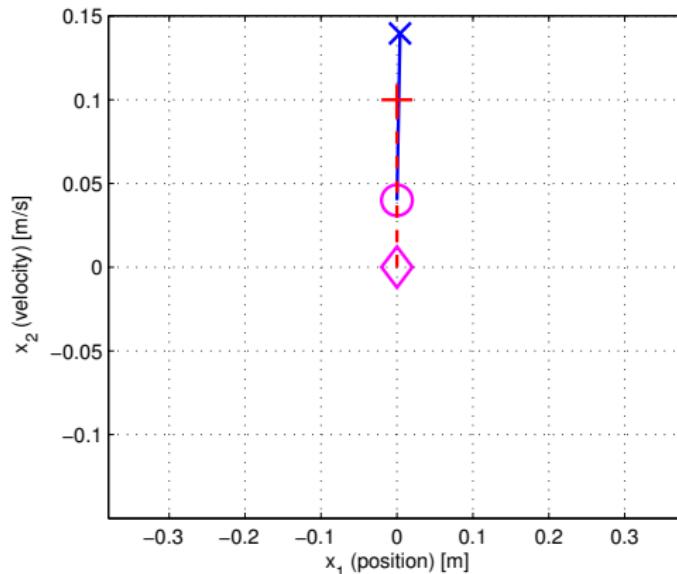
$\textcolor{blue}{x}$  versus  $\textcolor{red}{\hat{x}}$  at  $t = 100$

# Build intuition from simple systems

## State estimation via simulation

Naive estimate using *wrong* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \boxed{\hat{x}_0 \neq x_0}$$



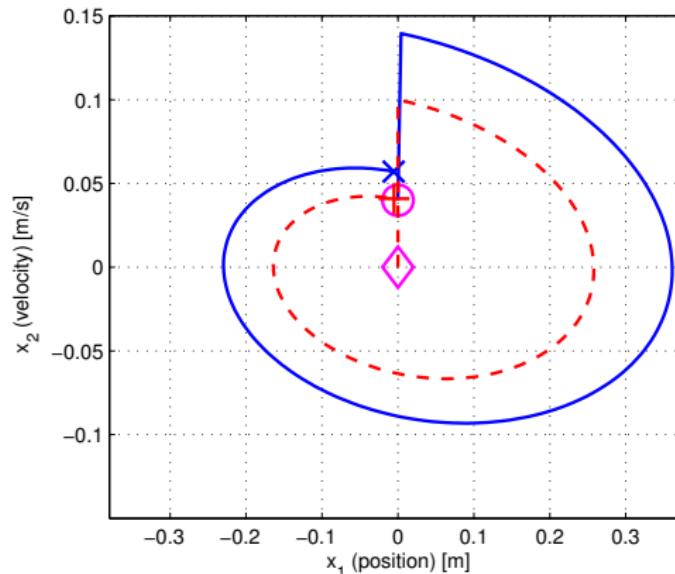
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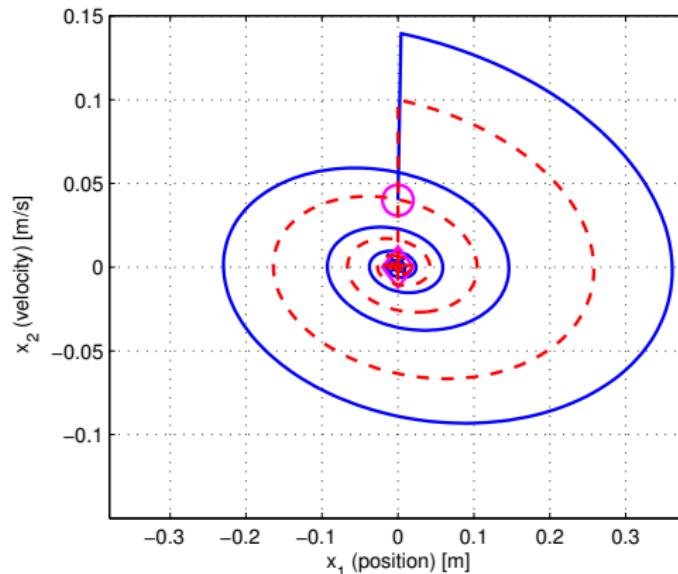
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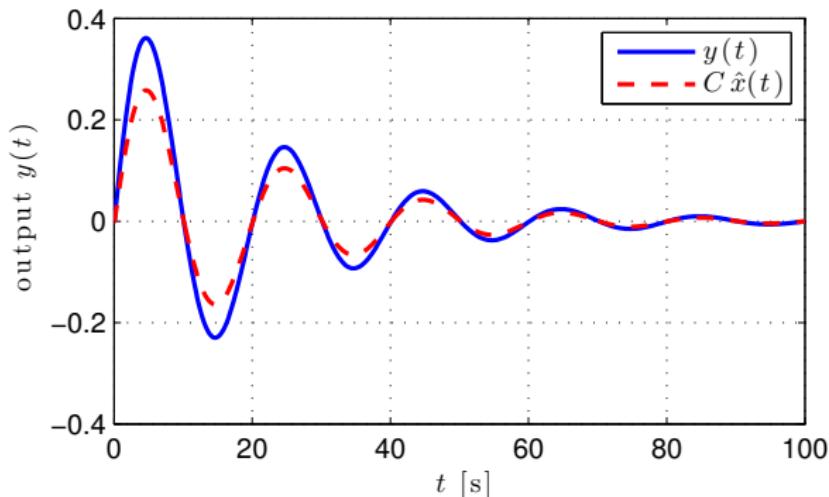
$\textcolor{blue}{x}$  versus  $\textcolor{red}{\hat{x}}$  at  $t = 100$

# Build intuition from simple systems

## State estimation via simulation

$\textcolor{blue}{x}$  och  $\hat{x}$  correspond to different outputs:

$$\textcolor{blue}{y} = C\textcolor{blue}{x} \quad \text{versus} \quad \hat{y} = C\hat{x}$$



# Estimating the states

## Correcting the state estimates

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- ▶ *Idea:* Feedback the prediction error  $y - C\hat{x}$  to correct  $\hat{x}$
- ▶ *Observer:* an estimator with a *correction term*

$$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K(y - C\hat{x})}_{\text{correction}}, \quad \hat{x}(0) = \hat{x}_0$$

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- ▶ Using matrix

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

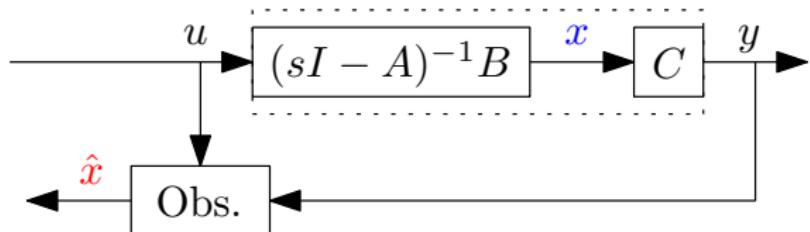
we can *design* the estimator.

# Estimating the states

## Correcting the state estimates

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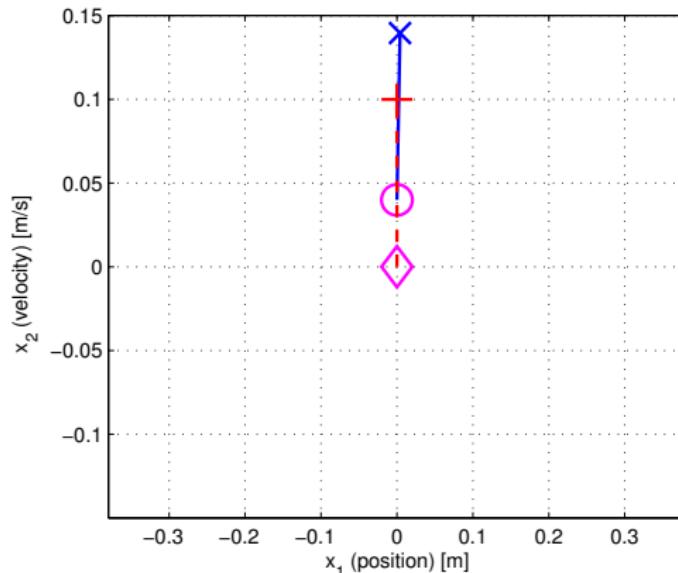


# Build intuition using simple systems

## State estimation using observer

Estimation using observer:

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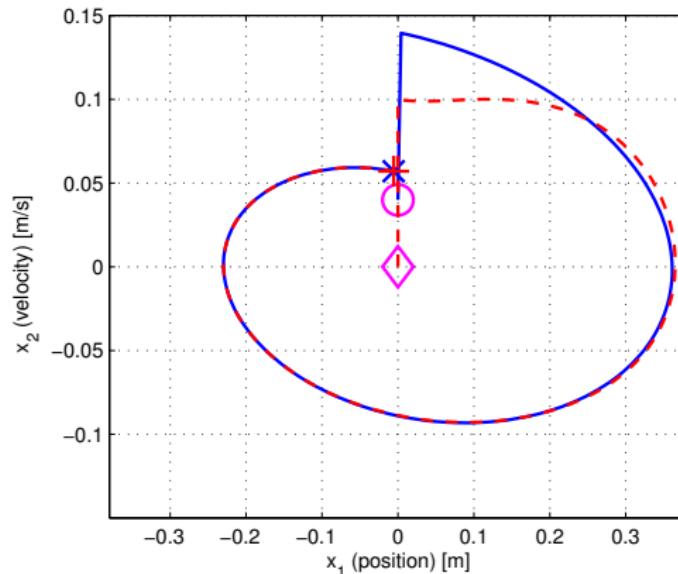
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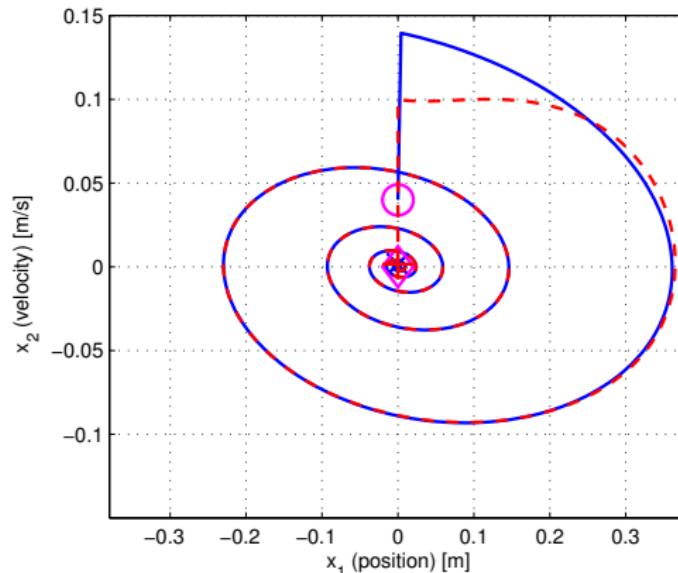
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$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}), \quad \hat{x}_0 \neq x_0$$



$x$  versus  $\hat{x}$  at  $t = 100$

# State estimation

## Estimation error and observability

---

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

[Board: derive evolution of estimation errors]

# State estimation

## Estimation error and observability

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

[Board: derive evolution of estimation errors]

### Result

Errors of observer described as system

$$\tilde{x}(t) = e^{(A - \mathbf{K}C)t} \tilde{x}(0)$$

and therefore  $\|\tilde{x}(t)\|$  decays at a rate

$$\max_i \operatorname{Re}\{\lambda_i\},$$

where  $\lambda_i$  denotes the  $i$ th eigenvalue of  $(A - \mathbf{K}C)$ .

# State estimation

## Estimation error and observability

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

### Result 9.2

State-space form is **observable** (cf.  $\det \mathcal{O} \neq 0$ )  $\Rightarrow$  matrix  $K$  can be chosen such that  $\tilde{x}$  vanish arbitrarily quick

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- $K$  is solved by polynomial  $\det(sI - A + KC) = 0$  with desired roots in left halfplane

# State estimation

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- ▶  $K$  is solved by polynomial  $\det(sI - A + KC) = 0$  with desired roots in left halfplane
- ▶ Quick observer  $\hat{x}$  is however sensitive to measurement noise!

# Feedback using estimated states

## Effect of estimation error

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State-space form

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

and controller form

$$u = -L\hat{x} + \ell_0 r$$

[Board: derive the closed-loop system with estimation error  $\tilde{x}$ ]

# Feedback using estimated states

## Effect of estimation error

Control using

$$u = -L\hat{x} + \ell_0 r$$

gives **closed-loop system**:

$$\begin{aligned}\dot{x} &= (A - BL)x + \underbrace{BL\tilde{x}}_{\text{effect of estimation error}} + B\ell_0 r \\ y &= Cx\end{aligned}$$

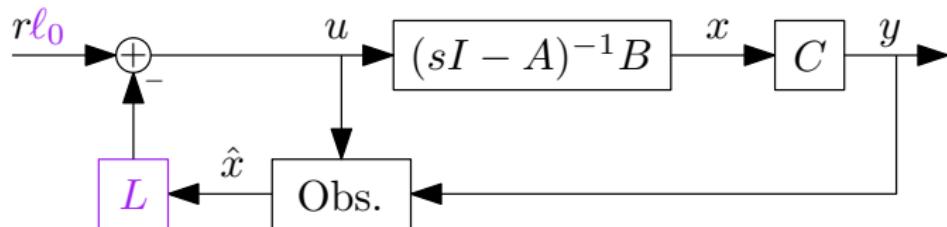
when

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

**Note:** when  $\tilde{x}$  vanishes  $\rightarrow 0$

# Feedback using estimated states

## The closed-loop system with observer



effect of estimation error

$$\begin{aligned}\dot{x} &= (A - BL)x + \overbrace{BL\tilde{x}}^{\text{effect of estimation error}} + B\ell_0 r \\ \dot{\tilde{x}} &= (A - KC)\tilde{x} \\ y &= Cx\end{aligned}$$

[Board: write the closed-loop system in state-space form]

# Feedback using estimated states

## The closed-loop system with observer

The closed-loop system with estimation error can be written as

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\bar{B}} \ell_0 r$$
$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x \\ \tilde{x} \end{bmatrix}}_{\bar{x}}$$

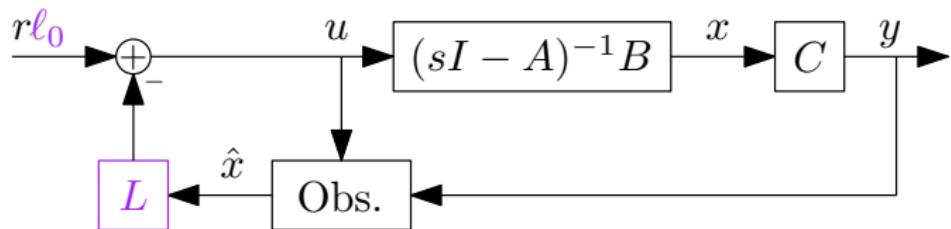
with extended state vector  $\bar{x}$ .

Thus

$$G_c(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}\ell_0$$

# Feedback using estimated states

## The closed-loop system with observer



Closed-loop system transfer function, ch.9.5 G&L

$Y(s) = G_c(s)R(s)$ , where

$$\begin{aligned}G_c(s) &= \bar{C}(sI - \bar{A})^{-1}\bar{B}\ell_0 \\&= C(sI - A + BL)^{-1}B\ell_0\end{aligned}$$

same poles as if states were known!

# Feedback using estimated states

## The closed-loop system with observer

---

Controller in  $\mathcal{L}$ :

$$u = -\textcolor{blue}{L}\hat{x} + \textcolor{blue}{\ell}_0 r \quad \Leftrightarrow \quad U(s) = -\textcolor{blue}{L}\hat{X}(s) + \textcolor{blue}{\ell}_0 R(s),$$

using  $\hat{x} = x - \tilde{x}$

# Feedback using estimated states

## The closed-loop system with observer

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using  $\hat{x} = x - \tilde{x}$ :

$$s\hat{X}(s) = (A - \textcolor{blue}{K}C - \textcolor{blue}{B}\textcolor{blue}{L})\hat{X}(s) + \textcolor{blue}{B}\textcolor{blue}{\ell}_0 R(s) + \textcolor{blue}{K}Y(s)$$

# Feedback using estimated states

## The closed-loop system with observer

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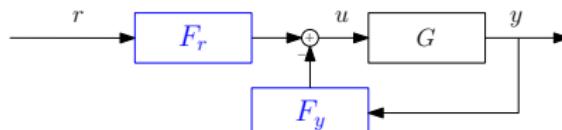
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General linear feedback form, ch.9.5 G&L

$\Rightarrow U(s) = \textcolor{blue}{F}_r(s)R(s) - \textcolor{blue}{F}_y(s)Y(s)$ , where

$$\textcolor{blue}{F}_r(s) = (1 - \textcolor{blue}{L}(sI - A + \textcolor{blue}{K}C + \textcolor{blue}{B}\textcolor{blue}{L})^{-1}\textcolor{blue}{B})\ell_0$$

$$\textcolor{blue}{F}_y(s) = \textcolor{blue}{L}(sI - A + \textcolor{blue}{K}C + \textcolor{blue}{B}\textcolor{blue}{L})^{-1}\textcolor{blue}{K}$$



# Summary and recap

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- ▶ Rules of thumb for pole placement
- ▶ Estimation using observer
- ▶ Feedback using estimated states
- ▶ Closed-loop system with observer