Systems and Control
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Introduction to computer control systems

Master program in embedded systems, period 2, 2011

Problem solving session III (Ex3) - Solutions

1. Use
$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{6} A^3 t^3 + \cdots$$
and show that
$$A^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & \lambda_n^k \end{bmatrix}.$$
Substitute A^k in the series to obtain
$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

2. (a) Use Kirchoff's (voltage and current) laws, and
$$v_L(t) = L \frac{di_L(t)}{dt}$$
$$v_R(t) = R\, i_R(t)$$
$$i_C(t) = C \frac{dv_C(t)}{dt}$$
where $v(t)$ denotes voltages and $i(t)$ currents, to obtain
$$\begin{aligned} \begin{bmatrix} \dot{i}_L(t) \\ \dot{v}_C(t) \end{bmatrix} &= \begin{bmatrix} -R_1/L & -1/L \\ 1/C & -1/(CR_2) \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} \end{aligned}$$

(b) $\Phi(t) = \begin{bmatrix} e^{-t}cos(t) & -e^{-t}sin(t) \\ e^{-t}sin(t) & e^{-t}cos(t) \end{bmatrix}$

(c) $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$

For a step changing to 0 V at time zero, the time response has only natural system response (due to initial conditions):

$$x(t) = \begin{bmatrix} e^{-t}(cos(t) - sin(t)) \\ e^{-t}(sin(t) + cos(t)) \end{bmatrix}$$

(d) Since for $t < 0$ $u(t)$ is the same as in (c), the system initial conditions are the same ($x(0)$), and hence the time response of the state-space variables is given by a sum of the result obtained in (c) with an extra term, due to the input (forced response):

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

with

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \frac{1}{2}e^{-t}\begin{bmatrix} e^t - (cos(t) - sin(t)) \\ e^t - (sin(t) + cos(t)) \end{bmatrix}$$

3. (a) $T = \begin{bmatrix} -1/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$ in $\hat{A} = TAT^{-1}$, where $\hat{A}$ is in diagonal form.

(b) $det(\lambda I - A) = \lambda(\lambda - 3) = \lambda^2 - 3\lambda \Rightarrow \hat{A} = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in the controllable canonical form.

$$T = S(\hat{A}, \hat{B})(S(A, B))^{-1} = \begin{bmatrix} 0.0909 & 0.1818 \\ 0.3636 & -0.2727 \end{bmatrix}$$

where $S(A, B)$ is the controllability matrix of the pair (A, B).

(c) $\hat{A} = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ and $\hat{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ in the observable canonical form.

$$T = (O(\hat{A}, \hat{C}))^{-1}O(A, C) = \begin{bmatrix} 5 & 6 \\ -4 & 4 \end{bmatrix}$$

where $O(A, C)$ is the observability matrix of the pair (A, C).

4. (a) The system is not controllable and it is unobservable.

(b) The system is not controllable but it is observable.