Introduction to computer control systems:
Selected exercises for the problem solving sessions
Master program in embedded systems, period 2, 2011

Problem solving session V (Ex5)

1. Consider the system composed by three-tanks in series [1] shown in Fig. 1. The inputs of the system are the tank 1 input flow $f_i(t)$ and the tank 1 output flow $f_0(t)$. The output of the system is the output flow $f_3(t)$ from the tank 3. $A_1$, $A_2$ and $A_3$ are the cross-sectional area of the tanks, $\rho$ is the density, and $h_1(t)$, $h_2(t)$ and $h_3(t)$ are the tank levels.

\begin{align*}
\rho A_1 \frac{dh_1(t)}{dt} &= \rho f_i(t) - \rho f_1(t) - \rho f_0(t) \\
\rho A_2 \frac{dh_2(t)}{dt} &= \rho f_1(t) - \rho f_2(t) \\
\rho A_3 \frac{dh_3(t)}{dt} &= \rho f_2(t) - \rho f_3(t) \\
f_1(t) &= C_{v1} \sqrt{h_1(t)} \\
f_2(t) &= C_{v2} \sqrt{h_2(t)} \\
f_3(t) &= C_{v3} \sqrt{h_3(t)}
\end{align*}

Figure 1: Three-tank process.
(a) Find the linear approximation in the state-space form at the equilibrium point $f_{i,0} = 5 \text{ m}^3/\text{h}$ and $f_{0,0} = 2 \text{ m}^3/\text{h}$. The model parameters are: $A_1 = 1.2 \text{ m}^2$, $A_2 = 1.5 \text{ m}^2$, $A_3 = 1 \text{ m}^2$, $C_{v_1} = 3.15$, $C_{v_2} = 2.8$ and $C_{v_3} = 2.5$.

(b) Compare the dynamic behaviour for the linear and nonlinear model by simulations for a step change of +10% and +30% in the input flow $f_i(t)$. Explain the differences.

(c) Compute the static gain for the state space model.

2. Consider the pendulum shown in Fig. 2. The system consists of a ball of mass $m$ located at the end of a massless rod with a length $l$. The moment of inertia of the pendulum about its pivot point is $J$, the viscous friction coefficient $B$ and the applied torque is $T$. The rotated angle $\theta$, which is the output variable and is taken as shown in Fig. 2.

![Figure 2: Pendulum.](image)

The angle $\theta$ is determined by

$$T = J \frac{d^2 \theta(t)}{dt^2} + B \frac{d\theta(t)}{dt} + mgl \sin(\theta(t))$$

This nonlinear differential equation of second order describes the dynamic behaviour of the pendulum. The model parameters are $l=1 \text{ m}$, $B=2 \text{ Nm/(rad/s)}$, $g=9.8 \text{ m/s}^2$, $m=3 \text{ kg}$ and $J = ml^2 \text{ kg m}^2$.

(a) Obtain the state space form at the equilibrium point $\theta_0 = 0$.
(b) Obtain the transfer function.
(c) Obtain the poles and zeros of the system.
(d) Analyse the response of the linear system to a sinusoidal signal $T = A \sin(\omega t)$ for:
   i. $A=0.5$, $\omega = 0.1 \text{ rad/s}$. 

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ii. $A=0.5$, $\omega = 0.04 \text{ rad/s}$.

iii. $A = 29.4$, $\omega = 0.04 \text{ rad/s}$.

(e) Compare by simulation the previous responses with the nonlinear system response. Explain the differences.

References