

Introduction to computer control systems
 Master program in embedded systems, period 2, 2011

Problem solving session VI (Ex6) - Solutions

1. System pulse-transfer operator: $H(q) = \frac{k}{q^2 + 0.4q + k} \Rightarrow$ the system transfer function is given by $G(s) = \frac{k}{s^2 + 0.4s + k}$. The roots of the polynomial $s^2 + 0.4s + k$ must lay inside the unit circle for the system to be stable. The system is hence stable if $0 < k < 0.04$. Note that k must be greater than zero for the feedback to be negative.

2. (a) $x_0 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

- (b) $\Delta x(k+1) = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 0 \end{bmatrix} \Delta x(k)$

- (c) $eig(A) = \{1.0916; -0.0916\}$. Since there is an eigenvalue λ_i such that $|\lambda_i| > 1$, the system is unstable.

3. (Based on Exercise 3.22 from [1])

NOTE: The (a) part of this question was changed from the one in the list of exercises. Here follows its correct version:

A dynamic system is given by a scalar differential equation with an algebraic expression given by

$$\begin{aligned} \frac{d}{dt}\xi &= -\xi + u\eta^3 \\ 0 &= -\eta + u^2e^\eta \end{aligned}$$

- (a) A control system should be designed to keep the system at a given stationary point ξ_0 . Determine the full operating point (ξ_0, u_0, η_0) when $\eta_0 = 1.1843$ and $u_0 > 0$.

The second equation gives $0 = -\eta_0 + u_0^2e^{\eta_0} \Leftrightarrow u_0 = \pm 0.6020$. Choose the positive solution for u_0 .

At the equilibrium point $\dot{\xi} = 0 \Leftrightarrow -\xi_0 + u_0\eta_0^3 = 0 \Leftrightarrow \xi_0 = 1$

- (b) The system input is u and its output is $y = \eta\xi$. Determine a linear state model, valid near the operating point determined in (a).

The second equation $0 = -\eta + u^2e^\eta$ gives $\eta = u^2e^\eta$. Replacing η in the first equation, it follows

$$\frac{d\xi}{dt} = -\xi + u^7e^{3\eta}$$

The linearization of the nonlinear term $f(u, \eta) = u^7e^{3\eta}$ around the stationary point (ξ_0, u_0, η_0) stands as

$$\begin{aligned} f(u, \eta) &\approx f(u_0, \eta_0) + \frac{\partial f}{\partial u}|_{(u_0, \eta_0)}(u - u_0) + \frac{\partial f}{\partial \eta}|_{(u_0, \eta_0)}(\eta - \eta_0) \\ u^7e^{3\eta} &\approx u_0^7e^{3\eta_0} + 7u_0^6e^{3\eta_0}(u - u_0) + 3u_0^7e^{3\eta_0}(\eta - \eta_0) \\ \frac{d\xi}{dt} + \xi &\approx \xi_0 + 7u_0^6e^{3\eta_0}(u - u_0) + 3u_0^7e^{3\eta_0}(\eta - \eta_0) \\ \frac{d\xi}{dt} &\approx -(\xi - \xi_0) + 7u_0^6e^{3\eta_0}(u - u_0) + 3u_0^7e^{3\eta_0}(\eta - \eta_0) \end{aligned}$$

By defining $E = \xi - \xi_0$, $U = u - u_0$ and $N = \eta - \eta_0$, and substituting u_0 and η_0 by its values, it follows that

$$\frac{dE}{dt} = -E + 11.6288U + 3N$$

In order to have a linear relation between η and u , $0 = -\eta + u^2e^\eta$ has to be linearized. By using Taylor series expansion around the equilibrium point,

$$\begin{aligned} \eta &\approx u_0e^{\eta_0} + 2u_0e^{\eta_0}(u - u_0) + u_0^2e^{\eta_0}(\eta - \eta_0) \\ \eta - \eta_0 &\approx 2u_0e^{\eta_0}U + u_0^2e^{\eta_0}N \\ N &\approx 2u_0e^{\eta_0}U + u_0^2e^{\eta_0}N \end{aligned}$$

Substituting u_0 and η_0 by its values, it follows that $N = -21.3511U$. The linearized differential equation is then given by

$$\frac{dE}{dt} = -E - 52.4244U$$

The linearized output is given by

$$\begin{aligned}
y - y_0 &\approx \xi_0(\eta - \eta_0) + \eta_0(\xi - \xi_0) \\
Y &= N + 1.1843E \\
Y &= -21.3511U + 1.1843E
\end{aligned}$$

$$\begin{aligned}
\dot{E} &= [-1]E + [-52.4244]U \\
Y &= [1.1843]E + [-21.3511]U
\end{aligned}$$

(c) How is the stability of the stationary operating point (ξ_0) ?

The linearized system has a pole in -1 so it is stable around the stationary point (ξ_0) .

4. (a) SS1: $\overline{c_A} = 0, \overline{c_B} = 0$.
SS2: $\overline{c_A} = 1/\beta, \overline{c_B} = 1/\alpha$.
M1:

$$\begin{aligned}
\dot{\Delta c} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Delta c \\
\Delta c_M &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \Delta c
\end{aligned}$$

M2:

$$\begin{aligned}
\dot{\Delta c} &= \begin{bmatrix} 0 & \alpha/\beta \\ \beta/\alpha & 0 \end{bmatrix} \Delta c \\
\Delta c_M &= \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \Delta c
\end{aligned}$$

- (b) M1 is asymptotically stable and M2 is unstable regardless the values of the process parameters α and β .

References

- [1] Mikael Johansson and Torsten Söderström. *Exercises Control Theory*. Uppsala University and Royal Institute of Technology, 2010.