Introduction to computer-controlled systems, 5 credits, 1RT485

Date and Time: 2011-08-16, 14-19

Place: Polacksbacken, Exam hall

Teacher on duty: Alexander Medvedev, tel. 3064.
Alexander will come to the exam hall to answer questions around 15.00.

Allowed aid: Course textbook (Glad, Ljung), calculator, copies of slides from the course, own notes from the course (in original or copy), mathematical handbook (Beta or similar).

Preliminary grade bounds: 3:[15, 20[, 4:[20, 25[, 5:[25, 30 = max]

NB: Please only one problem per sheet. Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Solutions have to be explained in detail and well-argumented.

GOOD LUCK!

1 Problems

1.1 Lab problem

a) (2 points)

The dynamics for the line tracking vehicle that was used in the process labs are given by a set of nonlinear equations. After simplification and linearization, they are defined as follows

$$\ddot{\phi}(t) = -\frac{1}{T}\dot{\phi}(t) + \frac{K}{T}u(t) \tag{1}$$

$$y(t) = L\phi(t) + v\int_0^t \phi(\tau)d\tau \tag{2}$$

where ϕ is the angular deviation from the track direction, u is the control signal, y is the measured position on the track and T, K, L are known constants.

Provide the state space description of the system with input u and output y using the states

$$x_1(t) = \int_0^t \phi(\tau)d\tau, \quad x_2(t) = \phi(t), \quad x_3(t) = \dot{\phi}(t)$$

b) (2 points)

Calculate the controllability matrix for the state space model found in a). Is the system controllable?

c) (2 points)

The transfer function for the system is given by

$$G(s) = \frac{K}{s^2(Ts+1)}$$

In time domain the PID regulator is given by

$$u(t) = K_p e(t) + K_I \int_0^t e(\tau)d\tau + K_D \frac{de(t)}{dt}$$

The closed loop system with reference r is shown in Fig. 1. Evaluate the transfer function for the closed loop system $G_c(s)$, i.e. the transfer function from r to y.

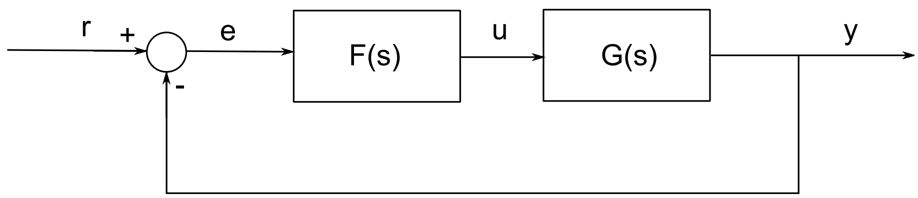

Figure 1: Closed loop system with regulator $F(s)$.

1.2 Multiple choice problem

This problem comprises six statements that can be agreed or disagreed with by answering yes or no. Each correct answer gives 0.5 points. A reasonable motivation to a correct answer adds 0.5 points. In total, provided all the questions are answered correctly and the answers are well motivated, this problem is worth 6 points.

1. The output of the discrete system given by the transfer function
$$W(z) = z^{-3}$$
can be unbounded for a bounded input signal.

2. The first-order differential equation
$$\dot{x} = x^2, \; x(0) \neq 0$$
has only unstable (diverging) solutions.

3. Same nonlinear system can exhibit both stable and unstable solutions, depending on the initial conditions.

4. The continuous dynamic system with a time delay described by the transfer function
$$W(s) = \frac{e^{-s}}{s+1}$$
has unit static gain.

5. Analog filtering of measurements and control signals is necessary in discrete control of analog plants.

6. A sine-wave of a curtain frequency applied as input to a nonlinear system produces a sine-wave response of the same frequency at the system output.

1.3 Problem A

Consider a nonlinear height tank, where the flow $F(t)$ is the input and the level $h(t)$ is the output. The system dynamics are described by

$$\frac{dh(t)}{dt} = \frac{F(t)}{A} - \frac{\beta}{A}\sqrt{h(t)}$$

β and A are parameters: $\beta = 1 \text{ m}^{2.5}/\text{min}$ and $A = 0.5 \text{ m}^2$.

a) (2 points)

Obtain the equilibrium point of the system for the input flow $F(t) = 2 \text{ m}^3/\text{min}$.

b) (2 points)

Obtain a linear state space representation of the system by linearization at the equilibrium point computed in the previous item with B as a unit matrix (i.e. $B = I$).

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

c) (2 points)

Obtain the impulse response, $y(t)$ (for $t > 0$), of the linearised continuous system obtained in the previous item, if a Dirac delta function is applied at the input u at $t_0 = 0$ and the initial condition is $x(0)$=0.1.

1.4 Problem B

Consider the continuous-time linear system given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (0 \quad 0.5) x\end{aligned}$$

a) (2 points)
Obtain the transfer function of the system.

b) (2 points)
Design a PID controller that yields the characteristic polynomial of the closed-loop system as $\Theta(s) = s^3 + 4s^2 + 6s + 4$. Obtain the values k_{P1}, k_{I1} and k_{D1}.

$$H_1(s) = k_{P1} + \frac{k_{I1}}{s} + k_{D1}s$$

c) (2 points)
Now consider another controller where only the proportional part is used with its proportional gain value is taken from **b)**

$$H_2(s) = k_{P2} = k_{P1}.$$

How does the steady state error of the closed-loop system change when this controller is used? Give motivation by comparing it with the PID controller from **b)**.

1.5 Problem C

Consider a continuous-time linear system given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu \\ y &= (1 \quad 0)x = Cx.\end{aligned}$$

a) (2 points)
Evaluate the transition matrix of the system, $\Phi(t) = \exp(At)$

b) (1 point) Is the continuous system controllable and observable? Motivate your answers with computations.

c) (2 points)
Sample the system with a sampling time T, and analyse observability of the sampled system. Is there a value of T that makes the sampled system unobservable?

d) (1 points)
For the sampled system, what eigenvalues would be suitable to assign to the matrix $\exp(AT) - LC$ by the observer gain L so that the estimation error of the observer

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

asymptotically converges to zero?

2 Solutions

2.1 Lab problem

a) The relation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \phi \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -\frac{1}{T}x_3 + \frac{K}{T}u \end{bmatrix}$$

gives the state space description

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K}{T} \end{bmatrix} u \\ y &= \begin{bmatrix} v & L & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

b) The controllability matrix is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{K}{T} \\ 0 & \frac{K}{T} & -\frac{K}{T^2} \\ \frac{K}{T} & -\frac{K}{T^2} & \frac{K}{T^3} \end{bmatrix}$$

clearly we have $rank(\mathbf{S}) = 3$ (i.e. full rank) and hence a controllable system.

c) The transfer function of PID regulator $F(s)$ could be derived by taking Laplace transform of the time domain expression under the assumption of zero initial conditions:

$$\begin{aligned} U(s) &= K_P E(s) + \frac{K_I}{s} E(s) + K_D E(s) s \\ F(s) &= \frac{U(s)}{E(s)} \\ &= \frac{K_D s^2 + K_P s + K_I}{s} \end{aligned}$$

From the block diagram, the relation in Laplace domain between reference signal $R(s)$ and output signal $Y(s)$ is given by

$$(1 + G(s)F(s))Y(s) = G(s)F(s)R(s)$$

The closed loop system $G_c(s)$ is then defined as

$$
\begin{aligned}
G_c(s) &= \frac{Y(s)}{R(s)} \\
&= \frac{G(s)F(s)}{1 + G(s)F(s)} \\
&= \frac{K(K_D s^2 + K_P s + K_I)}{s^3(Ts+1) + K(K_D s^2 + K_P s + K_I)}
\end{aligned}
$$

2.2 Multiple choice problem

1. No. The system is just a time delay for three steps. The output has exactly the same signal form as the input but with a time shift. Therefore, a bounded input yields a bounded output.

2. Yes. At any time, the velocity of x is positive which means that the variable can only increase.

3. Yes. Specific solutions of a nonlinear system are analyzed for stability. They might be stable or unstable, depending on where they originate from.

4. Yes. $W(s)|_{s=0} = 1$.

5. Yes. Analog filtering is typically used both at the input and at the output of a discrete controller. Filtering of the input is essential for anti-aliasing, filtering of the output does not let high-frequency content of rectangular control signal shape influence the dynamics of the plant.

6. No, nonlinear systems do not generally preserve the frequency of the input signal in the response. For instance, the static system $y = u^2$ doubles the frequency.

2.3 Problem A

a) Given the equations of the system

$$\frac{dh(t)}{dt} = \frac{F(t)}{A} - \frac{\beta}{A}\sqrt{h(t)}$$

From (1), by replacing $F(0) = F_0 = 2$ and $\frac{dh(t)}{dt} = 0$, the equilibrium point for the tank level is

$$h(0) = h_0 = 4 \text{ m}$$

b) Using Taylor series (until order 1) around the equilibrium point ($f_{i,0}$ and h_0), and assuming $u(t) = F(t) - F_0$ and $x(t) = h(t) - h_0$, the linear space-state representation at the equilibrium point is given by

$$\begin{aligned} \dot{x} &= \left[(\frac{\beta}{A})(-0.5h_0^{-1/2}) \right] x + [1]\, u \\ y &= \left[\frac{1}{A} \right] x \end{aligned}$$

By replacing the constant values, the linear space-state representation at the equilibrium point is given by

$$\begin{aligned} \dot{x} &= [-0.5]\, x + [1]\, u \\ y &= [2]\, x \end{aligned}$$

c) The output response, $y(t)$ (for $t > 0$) is computed as follows

$$\begin{aligned} y(t) &= x(t) \\ y(t) &= Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(\tau-t_0)}Bu(\tau)d\tau \\ y(t) &= 0.2e^{-0.5t} + 2e^{-0.5t} \\ y(t) &= 2.2e^{-0.5t} \qquad \text{for} \qquad \text{t} > 0 \end{aligned}$$

2.4 Problem B

a) The transfer function of the system is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ G(s) &= \frac{0.5}{s^2 + 3s + 2} \end{aligned}$$

b) The characteristic polynomial is given by $\Theta(s) = s^3 + 4s^2 + 6s + 4$. The closed-loop transfer function is given by

$$\begin{aligned} G_{CL1}(s) &= \frac{G(s)H_1(s)}{1 + G(s)H_1(s)} \\ G_{CL1}(s) &= \frac{0.5k_{D1}s^2 + 0.5k_{P1}s + 0.5k_{I1}}{s^3 + (3 + 0.5k_{D1})s^2 + (2 + 0.5k_{P1})s + 0.5k_{I1}} \end{aligned}$$

Matching the coefficients of both polynomials, we have $k_{P1} = 8$, $k_{I1} = 8$ and $k_{D1} = 2$.

c) The closed-loop transfer function using a P controller is given by

$$\begin{aligned} G_{CL2}(s) &= \frac{G(s)k_{P2}}{1 + G(s)k_{P2}} \\ &= \frac{0.5k_{P2}}{s^2 + 3s + 2 + 0.5k_{P2}} \end{aligned}$$

It is clear that in this case the steady-state error is different to zero, because the integral action is missing. Thus, $G_{CL1}(0) = 1$ and $G_{CL2}(0) \neq 1$.

2.5 Problem C

a) The transition matrix for is

$$\exp(At) = \begin{pmatrix} e^{-2t} & e^{-t} - e^{-2t} \\ 0 & e^{-t} \end{pmatrix}$$

b) The observability matrix of the continuous system is

$$\begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}.$$

It is non-singular and the system is observable. The controllability matrix of the continuous system is also non-singular which property guarantees controllability of the system:

$$\begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

c) The observability matrix of the sampled system is

$$\begin{pmatrix} C \\ C\exp(AT) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ e^{-2T} & e^{-T}(1 - e^{-T}) \end{pmatrix}.$$

It is non-singular whenever $e^{-T}(1 - e^{-T}) \neq 0$ which applies for any $T \neq 0$.

d) The matrix $\exp(AT) - CL$ is the system matrix of the state estimation error equation of the observer for the sampled system

$$e(t+1) = (\exp(AT) - CL)e(t)$$

and has to have all its eigenvalues within unit circle to yield a converging state estimation error.