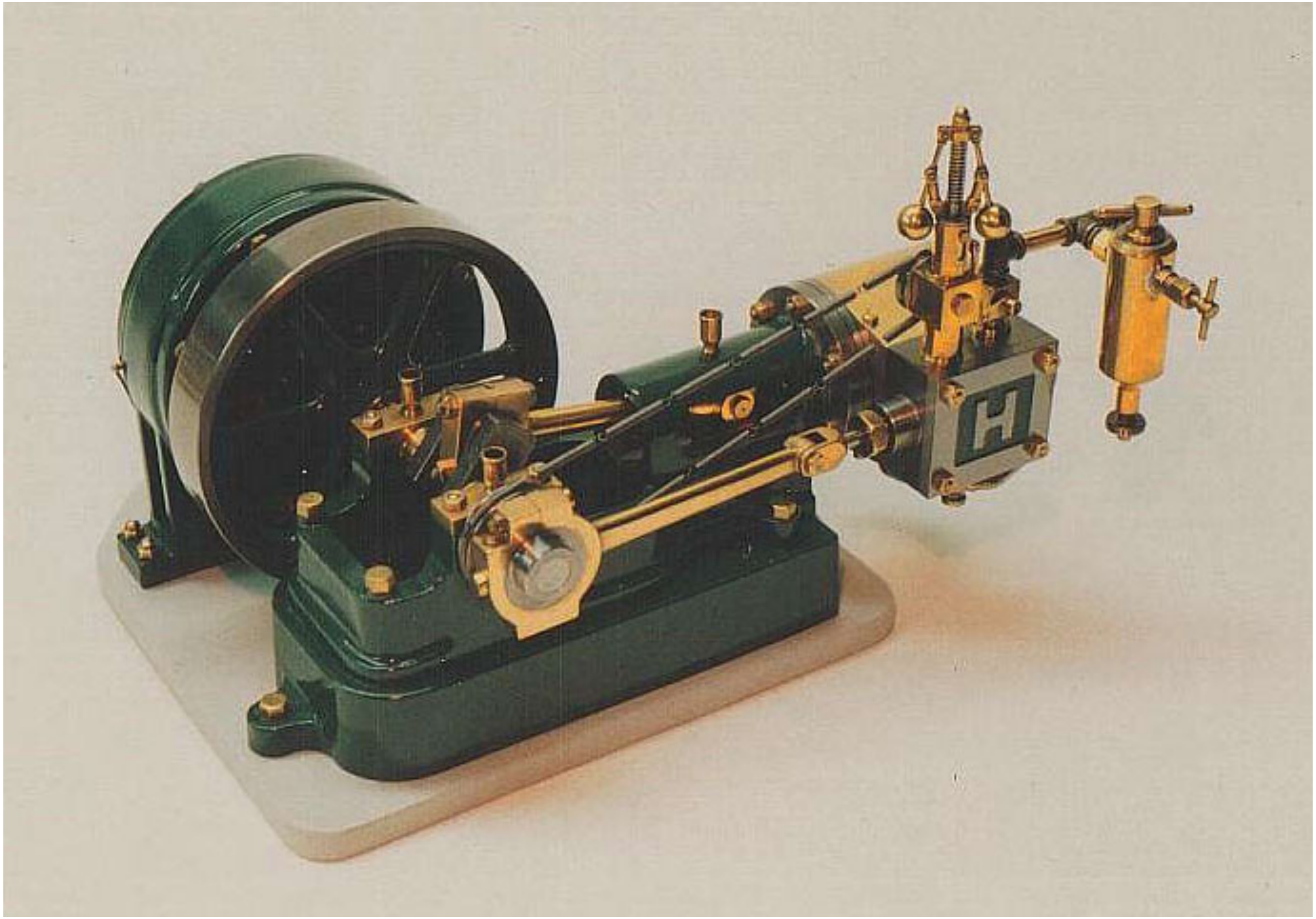
Lecture 2: Representation of linear systems in continuous and discrete-time

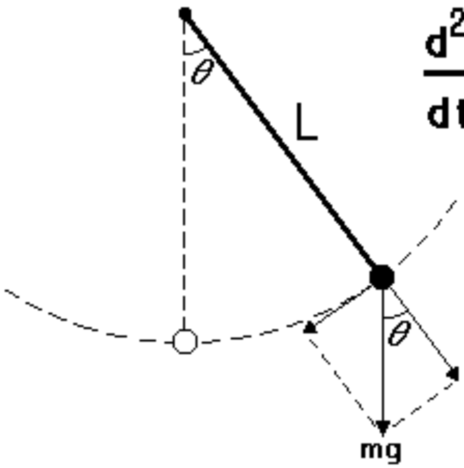
Outline

- Constants, variables, parameters
- Dynamic models
- State space models
- Impulse response


H

Constants, variables, parameters

- Constants: model variables that do not change with time
- System parameters: constants pertaining to system description
- Design parameters: constants that can be selected to give the systems desired properties.
- Variables (signals): model quantities that vary with time



$$\frac{d^2\theta}{dt^2} = -\frac{b}{m}\frac{d\theta}{dt} - \frac{g}{L}\sin\theta$$

θ is the angle the pendulum has moved from the vertical, **L** is the length of the pendulum, **g** is the acceleration due to gravity, **m** is the mass of the pendulum, and **b** is a damping coefficient.

Models for dynamic systems

- Dynamic system $\Leftrightarrow$ system with memory $\Rightarrow$ system's output signals depend on previous values of the input signal.
- Mathematic models for dynamic systems can be given by differential (continuous-time) or difference (discrete-time) equations whose solutions are functions of time.

$$g(y^{(n)}(t), y^{(n-1)}(t), \ldots, y(t), u^{(m)}(t), u^{(m-1)}(t), \ldots, u(t)) = 0$$

Input-output form (external form)

where *g(.)* can be a *linear or nonlinear* function, and, $y^{(k)}(t)$ and $u^{(k)}(t)$ are

- For continuous time

$$t \in [0, \infty), t \in (-\infty, \infty) \qquad y^{(k)}(t) = \frac{d^k}{dt^k} y(t) \qquad u^{(k)}(t) = \frac{d^k}{dt^k} u(t)$$

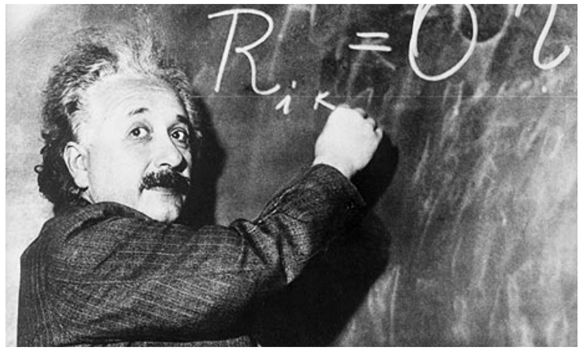
- For discrete time

$$t = 0, 1, 2, \ldots \qquad y^{(k)}(t) = y(t-k) \qquad u^{(k)}(t) = u(t-k)$$

Representation of linear systems

There are different ways to represent linear, time invariant and causal systems

- Input-output equations
- State space form
- Impulse response



Impulse response: discrete LTI systems

- The response of LTI systems to an arbitrary input is completely characterized by the impulse response

$$g(t)$$

- The output from a LTI system is the weighted sum of input values at all times

$$y(t) = \sum_{l=0}^{\infty} g(l)\,u(t-l)$$

Impulse response: continuous LTI systems

- The response of LTI systems to an arbitrary input is completely characterized by the impulse response

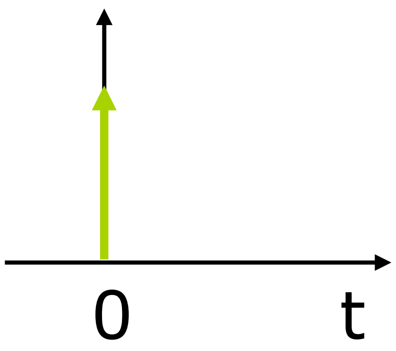
$$g(t)$$

- The output from a LTI system is the weighted sum of input values at all times

$$y(t) = \int_0^\infty g(\tau)\, u(t-\tau)\, d\tau$$

Impulse response: continuous LTI systems

- In continuous time, impulse response is the output signal of the system when the input signal is a Dirac delta function.



$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases}$$

Transfer Function: Transforms

- Exponential series:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots + \frac{1}{k!}x^k + \ldots$$

- Differentials:

$$pu(t) = \frac{du(t)}{dt}$$

- Taylor series:

$$u(t-\tau) = u(t) + \tau\frac{du(t)}{dt} + \frac{\tau^2}{2}\frac{d^2u(t)}{dt^2} + \ldots + \frac{\tau^k}{k!}\frac{d^ku(t)}{dt^k} + \ldots$$

- Euler:

$$e^{p\tau}u(t) = u(t) + \tau\frac{du(t)}{dt} + \frac{\tau^2}{2}\frac{d^2u(t)}{dt^2} + \ldots + \frac{\tau^k}{k!}\frac{d^ku(t)}{dt^k} + \ldots$$

Transfer Function: Transforms

- Laplace Transform:

$$U(s) = \int_0^\infty e^{-st} u(t)\,dt$$

- Discrete time *z*-transform:

$$U(z) = \sum_0^\infty e^{-zt} u(t)$$

- Convolution 2 Product:

$$y(t) = \int_0^\infty g(\tau)u(t-\tau)\,d\tau \longrightarrow Y(s) = U(s)G(s)$$

$$y(t) = \sum_0^\infty g(\tau)u(t-\tau) \longrightarrow Y(z) = U(z)G(z)$$

State space description (internal form) for continuous LTI systems

- Input-output form can be a differential equation of high order
- A "simpler" (but equivalent) description is a system of first order differential equations:

$x_i(t), i = 1,\dots,n$ internal variables (states);

$$x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$ state vector;

$$u = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$ input vector;

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

$$A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, D \in R^{p\times m}$$

If *A,B,C,D* are constant matrices then the model is linear and time-invariant

Solution of the state space equations for continuous LTI systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- Set initial conditions $x_0=x(t_0)$
- Set input signal $u(t)$, $t=[t_0,\infty)$

- The solution can be calculated analytically as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$ Convolution equation

- Matrix exponential $\Longrightarrow$ e^{At}

Solution of the state space equations for discrete LTI systems

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- Set initial conditions $x_0=x(t_0)$
- Set input signal $u(t)$, $t=t_0,t_0+1,\ldots$

- The solution can be calculated analytically as

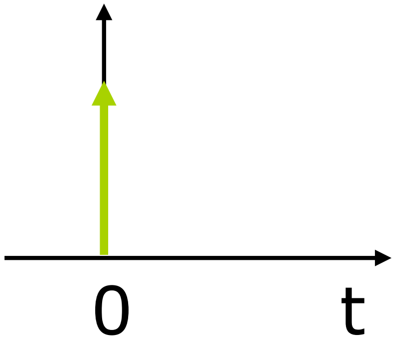
$$x(t) = A^{(t-t_0)}x(t_0) + \sum_{k=t_0}^{t-1} A^{t-k-1}Bu(k)$$
$$y(t) = Cx(t) + Du(t)$$

- Similarities: Continuous-time Discrete-time

$$e^{At} \Longrightarrow A^t$$
$$\int \Longrightarrow \sum$$

Impulse response: discrete LTI systems

- In discrete time, impulse response is the output signal of the system when the input signal is a Kronecker delta function.



$$\delta_k(t) = \begin{cases} 0 & t < 0 \\ 1 & t = 0 \\ 0 & t > 0 \end{cases}$$

For impulse response:

$$t_0 = 0$$
$$x_0 = 0$$
$$u(t) = \delta_k(t)$$

Impulse response: $y(t) = \begin{cases} D & for \quad t = 0 \\ CA^{t-1}B & for \quad t > 0, \quad t = 1,2,\ldots \end{cases} \Longrightarrow g(t)$

Conclusions Lecture 2

- LTI vs. Impulse Response Representation
- Transformations
- State Space Representations
- Continuous vs. Discrete time