Lecture 3: Model transformation and sampling

Outline

- Transfer operators and transfer functions
- From state space form to transfer function
- From transfer function to state space description
- State space transformation
- Sampling

Conclusions Lecture 2

- Causal LTI vs. Impulse Response Representation
- Transformations
- State Space Representations
- Continuous vs. Discrete time

ToDo:

- Find an article describing a control application
- What are here r,u,z,y,w,d,…
- Example: 2005 DARPA grand Challenge:

  r,u,z,y,n,w

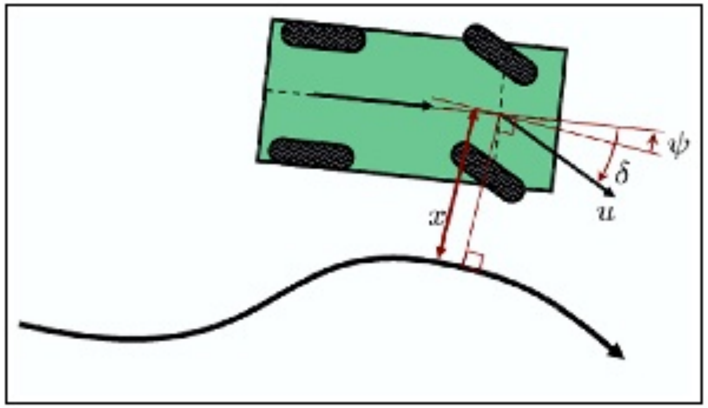
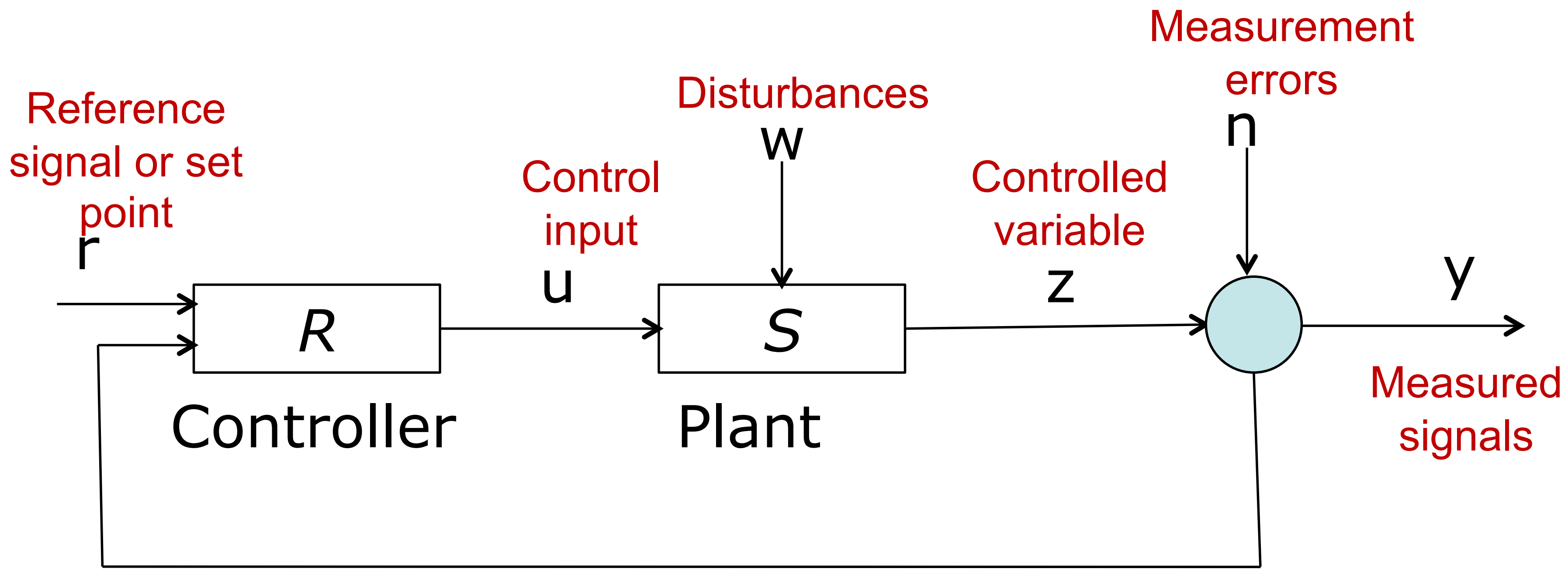


Figure 24: Illustration of the steering controller. With zero cross-track error, the basic implementation of the steering controller steers the front wheels parallel to the path. When cross-track error is perturbed from zero, it is nulled by commanding the steering according to a non-linear feedback function.

Lecture 1: Introduction and basic notions. The control problem



Transfer operator and transfer function

State space description for continuous time system

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

$A \in \mathcal{R}^{n\times n}, B \in \mathcal{R}^{n\times m}, C \in \mathcal{R}^{l\times n}, D \in \mathcal{R}^{l\times m}.$

Initial condition x(0)=0

Introduce the differentiation operator $p \equiv \frac{d}{dt}$

$$px(t) = Ax(t) + Bu(t)$$

$$(pI - A)x(t) = Bu(t)$$

$$x(t) = (pI - A)^{-1}Bu(t)$$

Laplace transform:

$$\mathcal{L}\{x(t)\} = X(s)$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s)$$

Formally: $p \rightarrow s$

the transfer operator

$$y(t) = W(p)u(t), W(p) = C(pI - A)^{-1}B + D$$

the transfer function

$$Y(s) = W(s)U(s), W(s) = C(sI - A)^{-1}B + D$$

Transfer operator and transfer function

State space description for discrete time system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Initial condition
x(0)=0

Introduce the forward shift operator $qx(t) \equiv x(t+1)$

$$qx(t) = Ax(t) + Bu(t)$$

$$(qI - A)x(t) = Bu(t)$$

$$x(t) = (qI - A)^{-1}Bu(t)$$

Z transform:

$$\mathcal{Z}\{x(t)\} = X(z)$$

$$\mathcal{Z}\{x(t+1)\} = zX(z)$$

Formally: $q \to z$

the transfer operator

$$y(t) = W(q)u(t), W(q) = C(qI - A)^{-1}B + D$$

the transfer function

$$Y(z) = W(z)U(z), W(z) = C(zI - A)^{-1}B + D$$

From external form to transfer function

A continuous system in input-output (external) form

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) =$$
$$b_0 u^{(k)}(t) + b_1 u^{(k-1)}(t) + \ldots + b_k u(t)$$
$$y^{(p)}(t) = \frac{d^p}{dt^p} y(t)$$

All the initial conditions are zero $\quad p \equiv \dfrac{d}{dt}$

$$(p^n + a_1 p^{n-1} + \cdots + a_n) y(t) = (b_0 p^k + b_1 p^{k-1} + \cdots + b_k) u(t)$$

Use the substitutions: $\quad p \to s \qquad y(t) \to Y(s), \ u(t) \to U(s)$

$$\left(s^n + a_1 s^{n-1} + \ldots + a_n\right) Y(s) = \left(b_0 s^k + b_1 s^{k-1} + \ldots + b_k\right) U(s)$$

$$W(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^k + b_1 s^{k-1} + \ldots + b_k}{s^n + a_1 s^{n-1} + \ldots + a_n}$$ The transfer function

Works also in reversed order, i.e. from a transfer function to the input-output form.

From external form to transfer function

A discrete system in input-output (external) form

$$y^{(n)}(t) + a_1 y^{(n-1)}(t-1) + \cdots + a_n y(t) =$$
$$b_0 u^{(k)}(t) + b_1 u^{(k-1)}(t-1) + \cdots + b_k u(t)$$

All the initial conditions are zero $\quad y^{(p)}(t) = y(t-p)$

In terms of $\quad qy(t) = y(t+1) \quad \left(q^{-1}y(t) = y(t-1)\right)$

$$(q^n + a_1 q^{n-1} + \cdots + a_n)y(t) = (b_0 q^k + b_1 q^{k-1} + \cdots + b_k)u(t)$$

Use the substitutions: $\quad q \to z,\ y(t) \to Y(z),\ u(t) \to U(z)$

$$(z^n + a_1 z^{n-1} + \cdots + a_n)Y(z) = (b_0 z^k + b_1 z^{k-1} + \cdots + b_k)U(z)$$

$$W(z) = \frac{Y(z)}{U(z)} = \frac{b_0 z^k + b_1 z^{k-1} + \cdots + b_k}{z^n + a_1 z^{n-1} + \cdots + a_n}$$ The transfer function

Choice of state variables

- State space equations carry information about internal system variables and how they are related to the input and output signals.
- For a physical system the state is composed of the variables required to account for storage of mass, momentum and energy.
- State variables in a mathematical model can be chosen arbitrarily.
- The dimension of the state vector is called the order of the system.
- The choice of state variables influences the accuracy in numerical integration (simulation) of the model.

State vector transformation

- State variables can be selected arbitrarily

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

- A new state vector $p = Tx,\ \det\ T \neq 0$

$$\begin{aligned} \dot{p}(t) &= TAT^{-1}p(t) + TBu(t) \\ y(t) &= CT^{-1}p(t) + Du(t) \end{aligned}$$

- State transformation has no effect on the transfer function of the system

$$W(s) = C(sI - A)^{-1}B + D$$

From transfer function to state space equation: method 1

$$W(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

Controllable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & \ldots & b_{n-1} - a_{n-1} b_0 & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

From transfer function to state space equation: method 2

$$W(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

Observable canonical form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & \ldots & 1 \\ -a_n & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ \vdots \\ b_{n-1} - a_{n-1} b_0 \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Sampling

Continuous system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Sampling time T

$$t = kT, k = 0,1,2,\ldots$$

Discrete system

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- The input signal is assumed to be piecewise constant

$$u(t) = u(kT), kT \le t < (k+1)T;$$

- The sampled system is given by: $x((k+1)T) = Fx(kT) + Gu(kT)$

$$y(kT) = Cx(kT) + Du(kT)$$

$$F = e^{AT}, G = \int_0^T e^{A\theta} B d\theta$$

- If A is invertible, then

$$G = A^{-1}\left(e^{AT} - I\right)B$$

The matrix exponential: e^{At} (the transition matrix)

- To find the sampled counterpart to a continuous sytem, it is necessary to calculate the matrix e^{At}
- A common way is $e^{At} = \mathscr{L}^{-1}(sI - A)^{-1}$
- For a diagonal A-matrix, we have

Diagonal form
$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \qquad e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$$

- In general, if the A-matrix is diagonalizable, we may consider a change of variables in the space state $p = Tx$, det $T \neq 0$

$$\begin{aligned} \dot{p}(t) &= TAT^{-1}p(t) + TBu(t) \\ y(t) &= CT^{-1}p(t) + Du(t) \end{aligned}$$

so that TAT^{-1} is diagonal and it simplifies the computation of e^{At}.

Summary

- State space description is used to describe large dynamic systems, possibly with several inputs and outputs.
- The state vector in a model can be assigned freely but the choice influences the structure of the model matrices and its numerical properties.
- Transfer operators are a “shorthand” way of describing systems in time domain
- Transfer functions describe systems in transform (Laplace and Z) domain
- Transfer functions are used for smaller systems under zero initial conditions and to emphasize the relationship between the input and output signal
- Canonic forms come in handy in the transformations from transfer functions to state space form