Lecture 5: Frequency domain properties

Outline

- Frequency response
- Poles and zeros
- Static gain
- First and second order systems

Frequency response

Continuous time state space description

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$A \in \mathcal{R}^{n\times n}, B \in \mathcal{R}^{n\times m}, C \in \mathcal{R}^{l\times n}, D \in \mathcal{R}^{l\times m}.$

Initial condition x(0)=0

Transfer function:

$$Y(s) = W(s)U(s), W(s) = C(sI - A)^{-1}B + D$$

Harmonic input signal: $W(s)|_{s=j\omega} = \mathrm{Re}\ W(j\omega) + j\mathrm{Im}\ W(j\omega)$

$$u(t) = a_0 \sin(\omega_0 t) \longrightarrow \boxed{W(s)} \longrightarrow y(t) = a_0 M(\omega_0) \sin(\omega_0 t + \varphi(\omega_0))$$

- Gain characteristics $M(\omega) = \sqrt{\mathrm{Im}^2 W(j\omega) + \mathrm{Re}^2 W(j\omega)}$
- Phase characteristics $\varphi(\omega) = \arctan \dfrac{\mathrm{Im} W(j\omega)}{\mathrm{Re} W(j\omega)}$

Frequency response

Discrete time state space description

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$A \in \mathcal{R}^{n\times n}, B \in \mathcal{R}^{n\times m}, C \in \mathcal{R}^{l\times n}, D \in \mathcal{R}^{l\times m}.$

Initial condition x(0)=0

Transfer function: $Y(z) = W(z)U(z), W(z) = C(zI - A)^{-1}B + D$

$$W(z)|_{z=e^{j\omega}} = \text{Re } W(e^{j\omega}) + j\text{Im } W(e^{j\omega})$$

$$e^{j\omega} = e^{j(\omega+2\pi)}$$

$$\omega < 2\pi$$

$$u(t) = a_0 \sin(\omega_0 t) \longrightarrow \boxed{W(z)} \longrightarrow y(t) = a_0 M(\omega_0) \sin(\omega_0 t + \varphi(\omega_0))$$

- Gain characteristics $M(\omega) = \sqrt{\text{Im}^2 W(e^{j\omega}) + \text{Re}^2 W(e^{j\omega})}$
- Phase characteristics $\varphi(\omega) = \arctan \dfrac{\text{Im} W(e^{j\omega})}{\text{Im} W(e^{j\omega})}$

Poles and zeros

- The transfer function (matrix-valued) for a state-space model

$$W(s) = C(sI - A)^{-1}B + D = \frac{1}{det(sI - A)}\left(Cadj(sI - A)^T B + Ddet(sI - A)\right)$$

- Poles are the roots to

$$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n = 0$$

i.e. the eigenvalues of A

- For a single-input single output system, the zeros of *W(s)* are the values of *s* such that *W(s)=0.*

- For a multi-input multi-output system, the zeros of *W(s)* are the poles of $W^{-1}(s)$.

- Zeros can be cancelled by poles

Static gain

- The static gain of a system is the finite (asymptotic) value of the system output *y* when the input *u* is constant with all elements equal to one.
- Static gain is not defined for unstable and marginally stable systems
- Continuous time:

$$W(s)|_{s=0} = C(sI - A)^{-1}B|_{s=0} = -CA^{-1}B$$

Final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

- Discrete time:

$$W(z)|_{z=1} = C(zI - A)^{-1}B|_{z=1} = C(I - A)^{-1}B$$

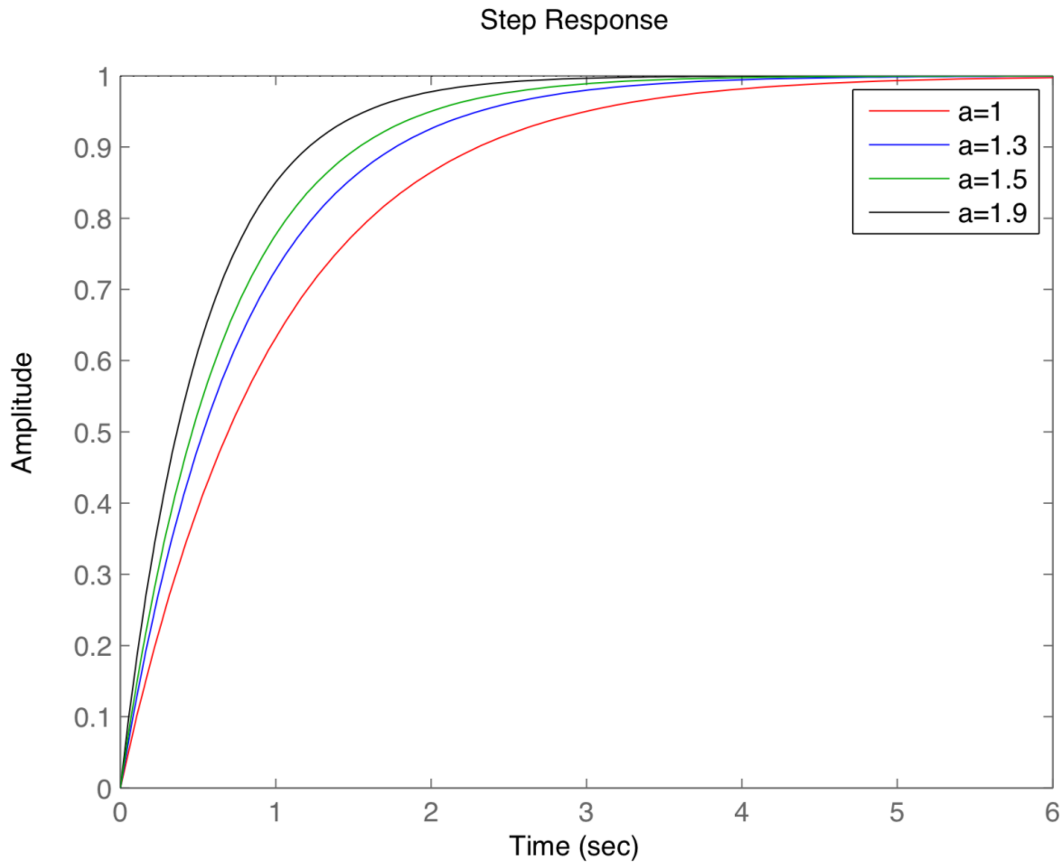
Final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)Y(z)$$

- Static gain can be established experimentally without a mathematical model

First order system: Step response

Single real pole: $$W(s) = \frac{a}{s+a}$$



- Static gain is one
- The greater the *a*, the faster the system
- A static system is infinitely fast

Second order system: Step response

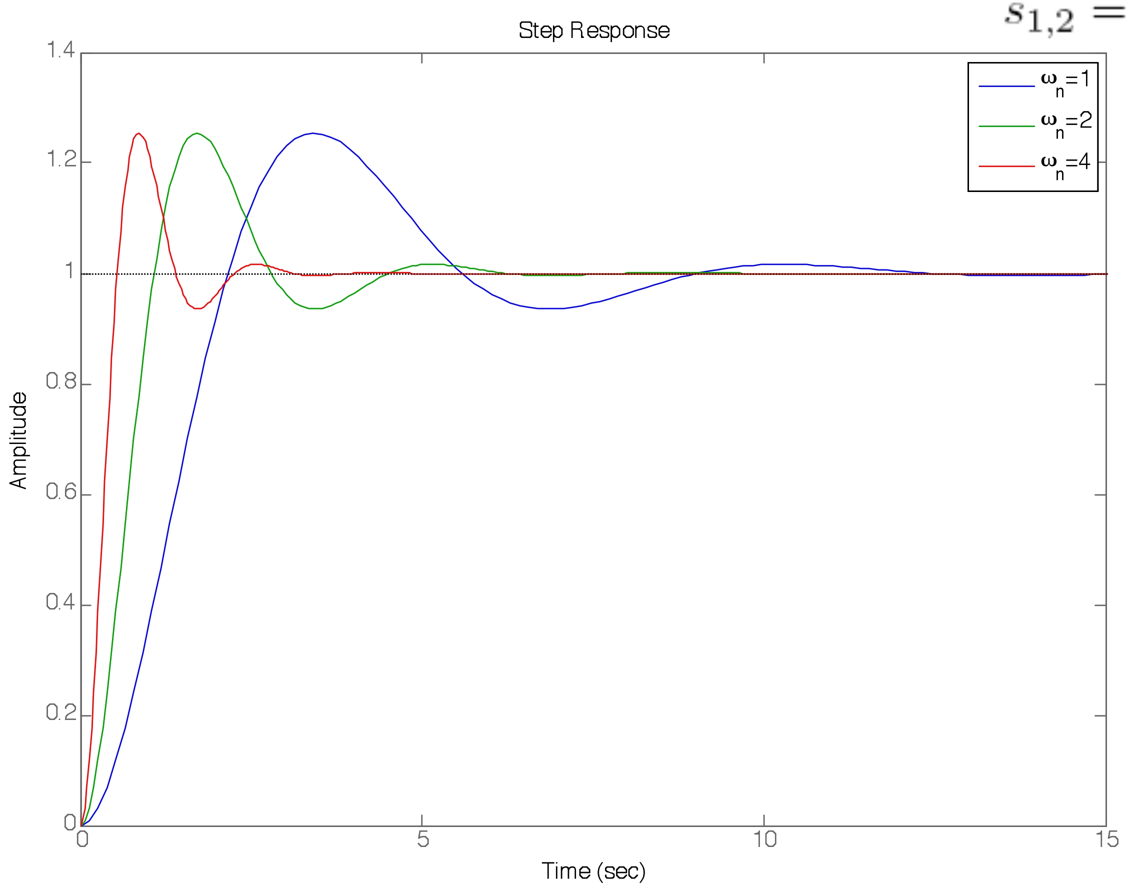
Transfer function:

$$W(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

ω_n – resonance frequency

ξ - damping factor

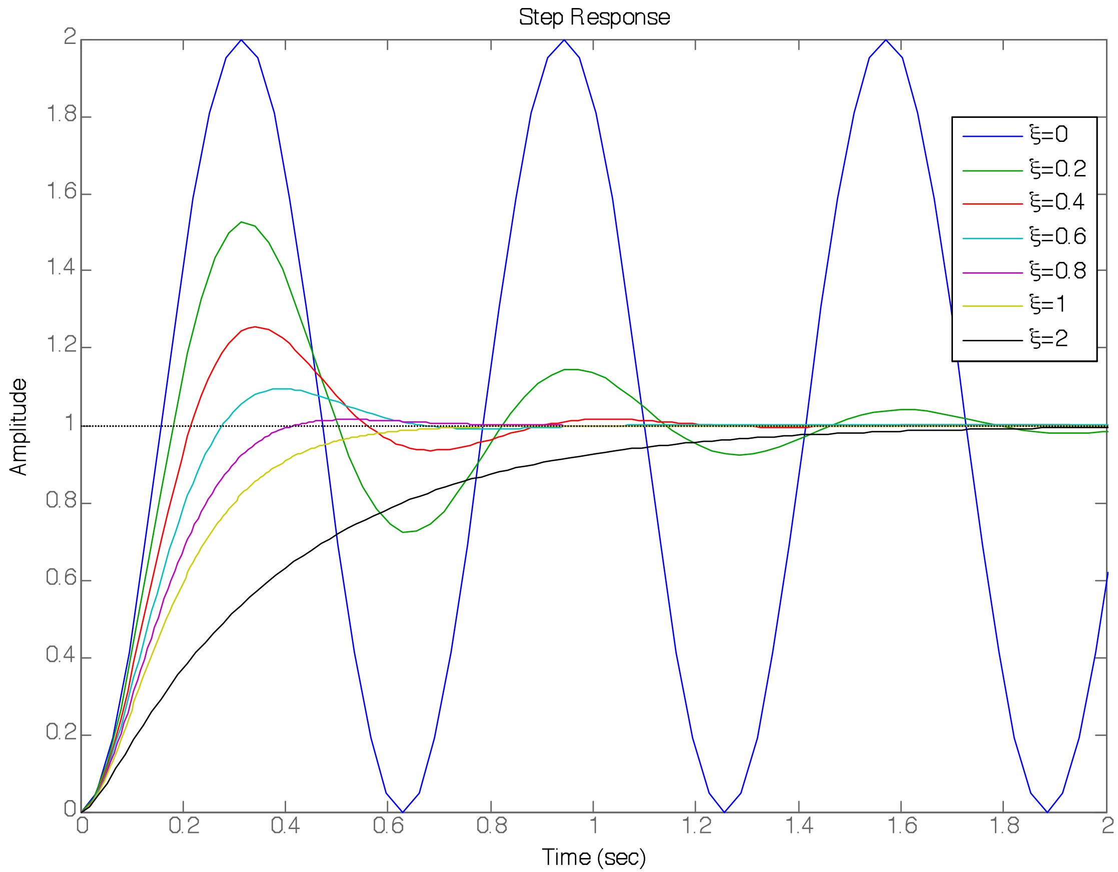
$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$



- Static gain is 1.
- The higher the resonance frequency, the faster the oscillations
- ξ =0.4

Second order system: Step response

$$W(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



- The higher the damping factor is, the lesser the oscillations
- ξ =0 corresponds to undamped oscillations
- ξ =1 corresponds to a critically damped system
- 0<ξ<1 corresponds to an underdamped system
- ξ>1 corresponds to an overdamped system
- ω_n=10

Summary

- Frequency response of a system characterizes its reaction to a harmonic (sinus-like) input and is easiest evaluated from the transfer function of the system
- Zeros of a transfer function specify what frequencies are not reproduced by the system
- Poles of a transfer function are the eigenvalues of the matrix A. They, together with the zeros, determine the system performance.
- The definitions of poles and zeros, as well as the calculation of them, are valid also for discrete time systems ($s \rightarrow z$)