Lecture 6: LTI system response to inputs. Controllability and observability

Outline

- System response to inputs
- Observability
- Controllability

System response to inputs for continuous LTI systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- Set initial conditions $x_0=0$
- Set input signal $u(t)$, $t=[t_0,\infty)$

- The output of the system, by assuming $x_0=0$, is given by

$$y(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

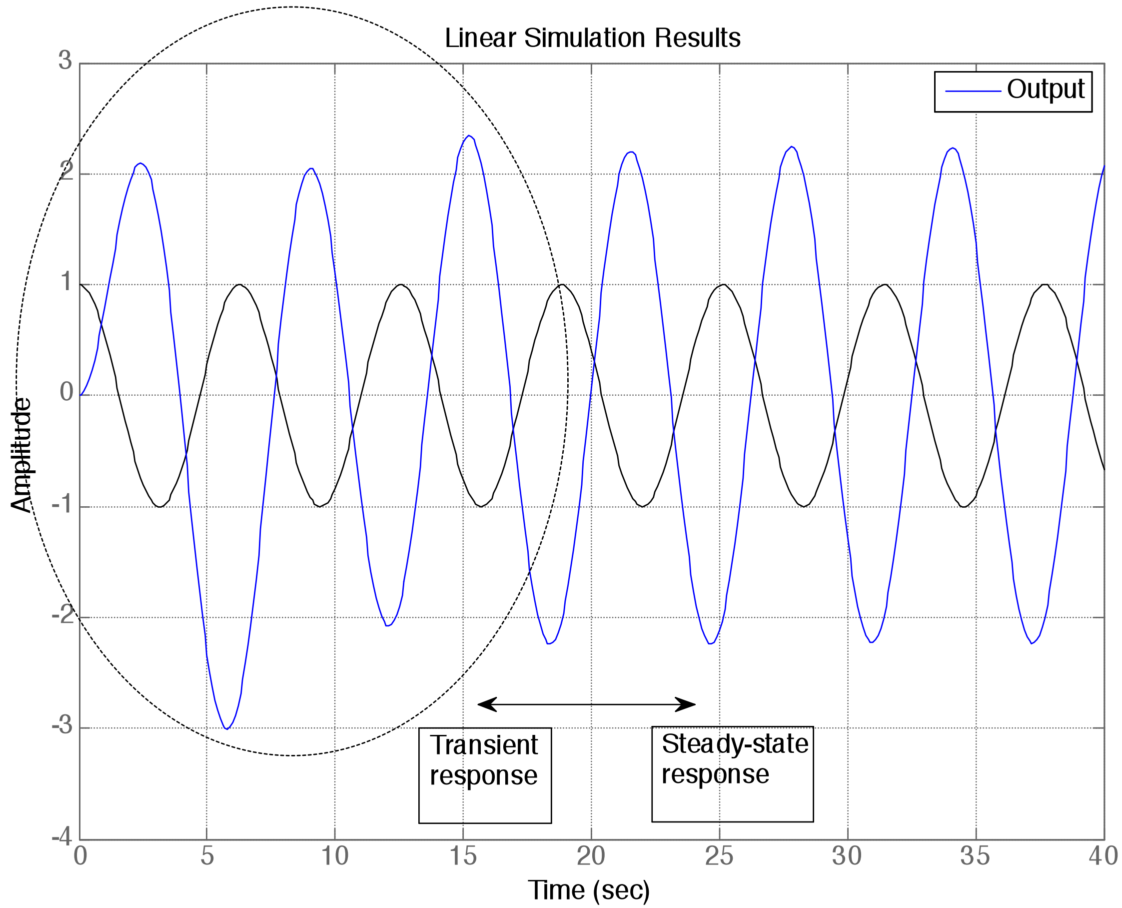
- The response to the inputs consists of two components:
  - The transient response
  - The steady-state response

Example: Periodic input

$$\dot{x}(t) = \begin{bmatrix} -0.4 & -0.2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 2 \end{bmatrix} x(t)$$

$$u(t) = \cos(t)$$



Step response for continuous LTI systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- Set initial conditions $x_0=0$
- Set input signal $u(t)$, $t=[t_0,\infty)$

- The output of the system, by assuming $x_0=0$ and a unit step as input, is given by

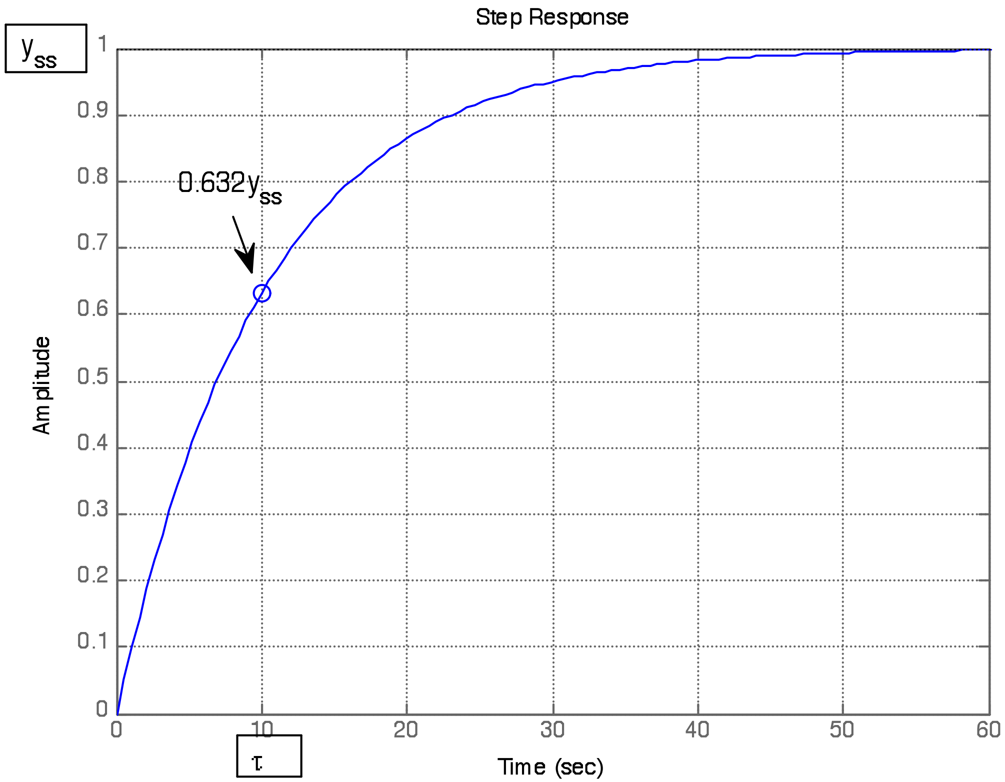
$$y(t) = \int_{t_0}^{t} Ce^{A(t-\tau)}Bd\tau + D$$

$$y(t) = \underbrace{CA^{-1}e^{At}B}_{\text{Transient response}} + \underbrace{D - CA^{-1}B}_{\text{Steady-state response}} \quad t > 0$$

First order system: Step response

- The time constant of a system provides useful information about the dynamics of a process and also for the choice of the sampling time

$$W(s) = \frac{K}{\tau s + 1}$$



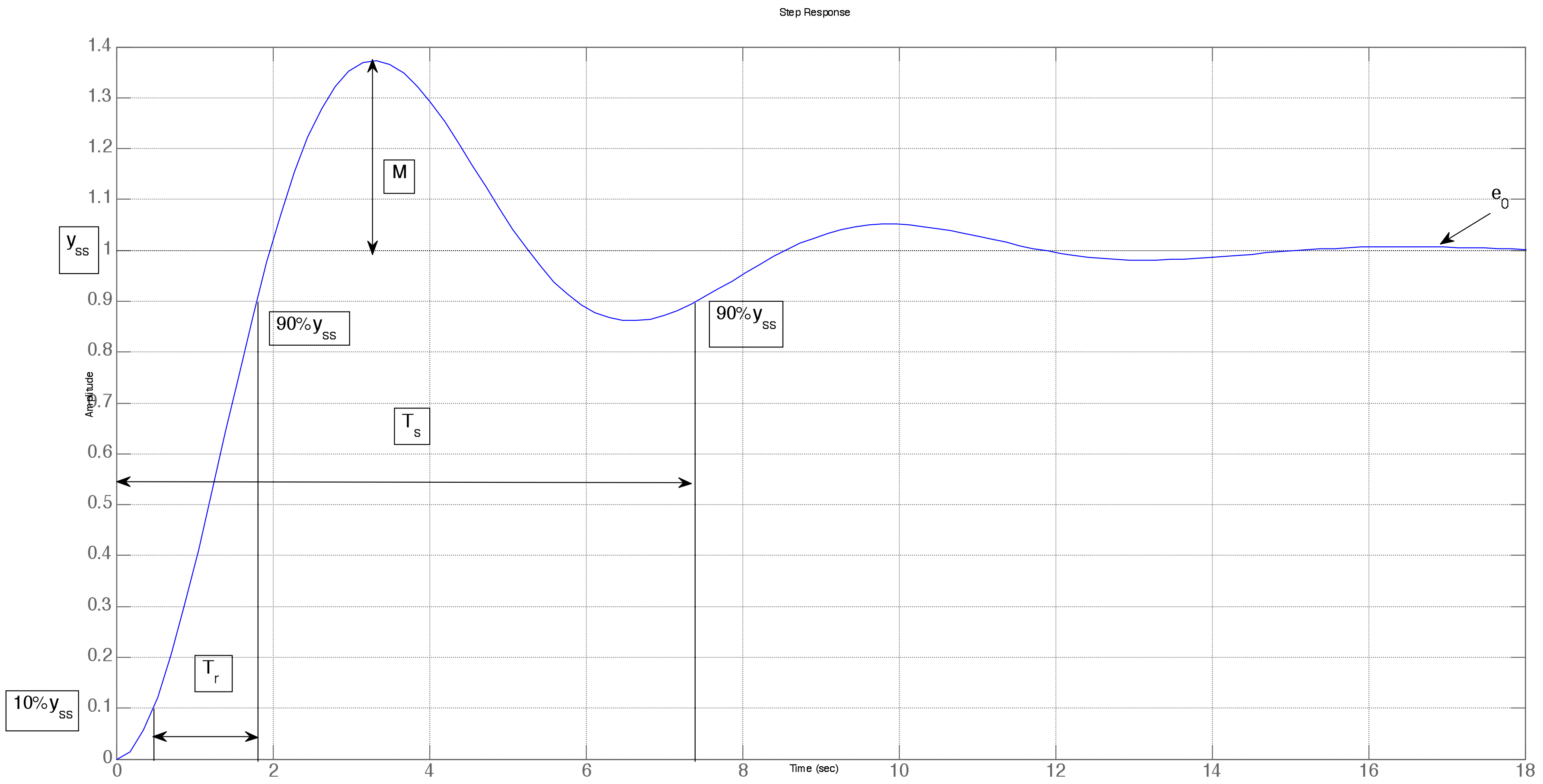
- $K=1$
- y_{ss} is the steady-state value
- τ is the time constant
- $y(\tau)=63.2\% yss$

$$t = \tau$$

$$y(\tau) = (1 - e^{-1}) y_{ss}$$

Step response for underdamped systems

- y_{ss} is the steady-state value, M is the overshoot, e_0 is the static error, T_r is the rise time and T_s is the settling time.



Observability of LTI systems

Continuous time LTI

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

Discrete time LTI

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

- Let $u(t)=0,\ t\geq 0$ and $x(0)=x^*,\ x^*\neq 0$. The state x^* is said to be unobservable if $y(t)=0$ for all $t\geq 0$.
- The system is said to be observable if it lacks unobservable states.
- The system is observable if and only if $\operatorname{rank} \mathcal{O}(A,C) = n$

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\mathcal{O}(A,C)$ is the observability matrix and is the same for the continuous and discrete case

- Any system in observability canonical form is observable

Controllability of LTI systems

- The state x^* is said to be controllable if there is an input that in **finite time** drives the system state vector to x^* from the initial state $x(0)=0$
- The system is controllable if all states are controllable
- The system is controllable if and only if $\operatorname{rank} \mathcal{S}(A,B) = n$

$$\mathcal{S}(A,B) = \left[\begin{array}{ccc} B & AB & \dots A^{n-1}B \end{array} \right]$$

$\mathcal{S}(A,B)$ is the controllability matrix and is the same for the continuous and discrete case

- Any system in controllability canonical form is controllable

Summary

- Transient response shows the discrepancy between the current conditions of the system and the steady-state solution after the inputs are applied
- Steady-state response shows the long term behavior of the system for the applied inputs
- Step response provides useful information about the system
- Control design criterion is frequently defined by using specifications of the step response
- Controllability and observability are important prerequisites for controller design