Lecture 6: LTI system response to inputs.  
Controllability and observability

Outline

- System response to inputs
- Observability
- Controllability
System response to inputs for continuous LTI systems

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

- Set initial conditions \( x_0 = 0 \)
- Set input signal \( u(t), \, t=[t_0, \infty) \)

The output of the system, by assuming \( x_0 = 0 \), is given by

\[ y(t) = \int_{t_0}^{t} Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t) \]

The response to the inputs consists of two components:
- The transient response
- The steady-state response
Example: Periodic input

\[
\dot{x}(t) = \begin{bmatrix} -0.4 & -0.2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)
\]

\[
y(t) = \begin{bmatrix} 0 & 2 \end{bmatrix} x(t)
\]

\[u(t) = \cos(t)\]
Step response for continuous LTI systems

\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ y(t) = Cx(t) + Du(t) \]

- Set initial conditions \( x_0 = 0 \)
- Set input signal \( u(t), t = [t_0, \infty) \)

The output of the system, by assuming \( x_0 = 0 \) and a unit step as input, is given by

\[
y(t) = \int_{t_0}^{t} Ce^{A(t-\tau)} B d\tau + D
\]

\[
y(t) = CA^{-1}e^{At} B + D - CA^{-1}B \quad t > 0
\]

Transient response \hspace{1cm} \text{Steady-state response}
First order system: Step response

- The time constant of a system provides useful information about the dynamics of a process and also for the choice of the sampling time

\[ W(s) = \frac{K}{\tau s + 1} \]

- \( K = 1 \)
- \( y_{ss} \) is the steady-state value
- \( \tau \) is the time constant
- \( y(\tau) = 63.2\% y_{ss} \)

\[ t = \tau \]
\[ y(\tau) = (1 - e^{-1}) y_{ss} \]
Step response for underdamped systems

- $y_{ss}$ is the steady-state value, $M$ is the overshoot, $e_0$ is the static error, $T_r$ is the rise time and $T_s$ is the settling time.
# Observability of LTI systems

### Continuous time LTI

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]

### Discrete time LTI

\[
\begin{align*}
x(t + 1) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]

- Let \( u(t) = 0, \ t \geq 0 \) and \( x(0) = x^*, \ x^* \neq 0 \). The state \( x^* \) is said to be unobservable if \( y(t) = 0 \) for all \( t \geq 0 \).
- The system is said to be observable if it lacks unobservable states.
- The system is observable if and only if
  \[\text{rank } \mathcal{O}(A, C) = n\]

\[
\mathcal{O}(A, C) = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

- \( \mathcal{O}(A, C) \) is the observability matrix and is the same for the continuous and discrete case.
- Any system in observability canonical form is observable.
Controllability of LTI systems

• The state \( x^* \) is said to be controllable if there is an input that in finite time drives the system state vector to \( x^* \) from the initial state \( x(0) = 0 \)
• The system is controllable if all states are controllable
• The system is controllable if and only if \( \text{rank } S(A, B) = n \)

\[
S(A, B) = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}
\]

\( S(A, B) \) is the controllability matrix and is the same for the continuous and discrete case

• Any system in controllability canonical form is controllable
Summary

- Transient response shows the discrepancy between the current conditions of the system and the steady-state solution after the inputs are applied.
- Steady-state response shows the long term behavior of the system for the applied inputs.
- Step response provides useful information about the system.
- Control design criterion is frequently defined by using specifications of the step response.
- Controllability and observability are important prerequisites for controller design.