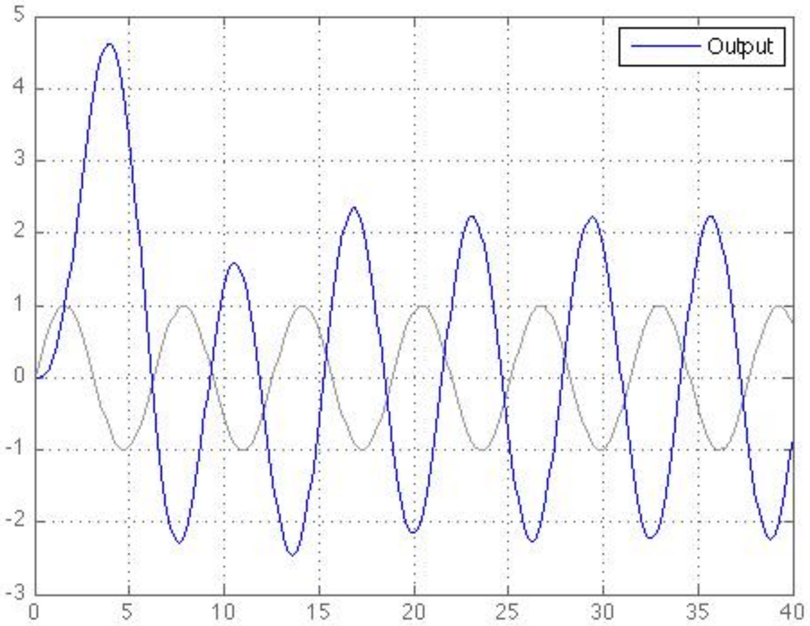
Lecture 7: Stability

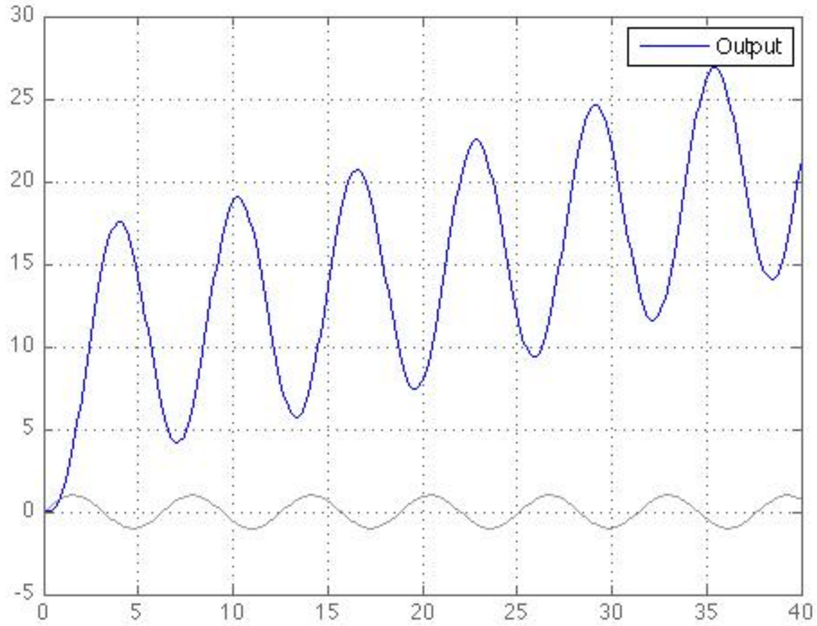
Outline

- Bounded-input Bounded-output (BIBO) stability
- Stability of solutions
- Stability for linear time invariant systems
- The Nyquist stability criterion
- Stability for equilibria via linearization

BIBO stability

- Many different stability notions exist.
- A general system is BIBO stable, if a bounded input leads to a bounded output.
- A system is input-output stable if it has finite gain (!).





Stability of solutions

- Here stability is understood as stability of solutions of system equations with respect to initial conditions.
- For a general system, stability depends on where in the state space the state vector of the system is.

$$\dot{x}(t) = f(x(t))$$
$$y(t) = h(x(t))$$
$$x(0) = x_0$$

$$x(t+1) = f(x(t))$$
$$y(t) = h(x(t))$$
$$x(0) = x_0$$

- The solution x*(t), for the initial state x*(0), is said to be stable if for each ε>0 there is a δ such that $\left|x^*(0) - x(0)\right| < \delta$ implies that $\left|x^*(t) - x(t)\right| < \varepsilon$ for all t>0.
- The solution x*(t) is said to be asymptotically stable if it is stable and there exist a δ such that $\left|x^*(0) - x(0)\right| < \delta$ implies that $\left|x^*(t) - x(t)\right| \to 0$ as t tends to ∞.

Stability for LTI systems

- LTI systems: stability is a system property (defined by systems parameters) and applies notwithstanding the initial conditions

$$\dot{x}(t) = Ax(t) \qquad\qquad x(t+1) = Ax(t)$$

All parameters are in the matrix A

- Initial conditions response

$$x(t) = \exp(At)x_0,\ t \in [1, \infty) \qquad\qquad x(t) = A^t x_0,\ t = 0, 1, 2, \ldots$$

- The difference between two solutions with different initial values (and the same input) is given by

$$x^*(t) - x(t) = e^{A(t-t_0)}\left(x^*(t_0) - x(t_0)\right)$$

It is determined by the properties of *A* (here we refer to unstable, stable or asymptotically stable systems)

- The behavior of e^{At} and A^t is related to the eigenvalues of the matrix *A*.

Stability for LTI systems

- Initial conditions response

$$x(t) = \exp(At)x_0, \ t \in [1, \infty) \qquad x(t) = A^t x_0, \ t = 0, 1, 2, \ldots$$

- Eigenvalues and eigenvectors of *A*: $A\xi_i = \lambda_i \xi_i, \ i = 1, \ldots, n$
- Consider the case of single and distinct eigenvalues:

$$A \begin{bmatrix} \xi_1 & \ldots & \xi_n \end{bmatrix} = \text{diag}(\lambda_1, \ldots, \lambda_n) \begin{bmatrix} \xi_1 & \ldots & \xi_n \end{bmatrix}$$

- Introduce the transformation matrix: $T = \begin{bmatrix} \xi_1 & \ldots & \xi_n \end{bmatrix}$
- For normalized eigenvectors *T* is unitary, i.e. $T^T T = I$

$$T^{-1}AT = T^T AT = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

- Let *x*=*Tz*, z –new state vector, $x_0 = Tz_0$

$$\dot{z} = \text{diag}(\lambda_1, \ldots, \lambda_n) z \qquad z(t+1) = \text{diag}(\lambda_1, \ldots, \lambda_n) z(t)$$

$$z(t) = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) z_0 \qquad z(t) = \text{diag}(\lambda_1^t, \ldots, \lambda_n^t) z_0$$

Stability for LTI systems

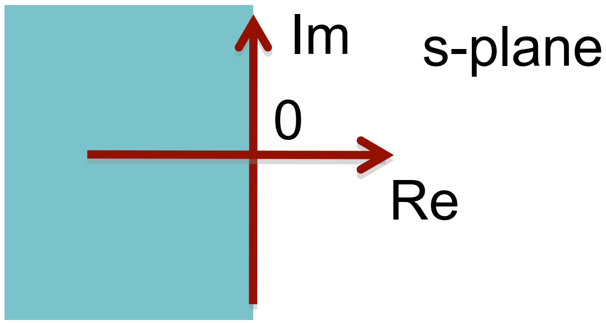
Continuous system:

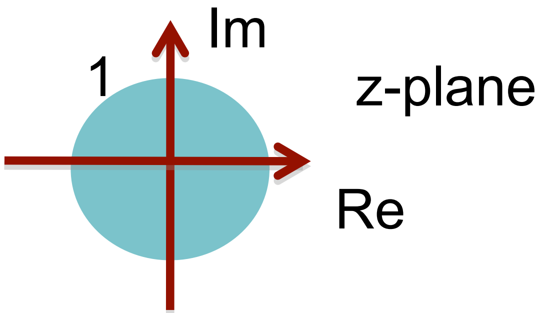
$$z(t) = \text{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) z_0$$

Discrete system:

$$z(t) = \text{diag}(\lambda_1^t, \dots, \lambda_n^t) z_0$$

- An LTI system is asymptotically stable if and only if all eigenvalues of the matrix A are inside the stability region.





- If an eigenvalue λ_i is outside of the stability region then the system is unstable
- If all eigenvalues are inside the stability region or on the stability border and those that are on the stability border are single, then the system is marginally stable. The system output can though be unbounded despite bounded input.

Stability for continuous LTI system

- LTI – linear time-invariant system (model)
  - Input-output form $\quad a_1, \ldots, a_n, b_0, \ldots, b_k = \text{const}$

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y(t) =$$
$$b_0 u^{(k)}(t) + b_1 u^{(k-1)}(t) + \ldots + b_k u(t)$$
$$y^{(p)}(t) = \frac{d^p}{dt^p} y(t) \qquad W(s) = \frac{b_0 s^k + b_1 s^{k-1} + \ldots + b_k}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

  - State space form

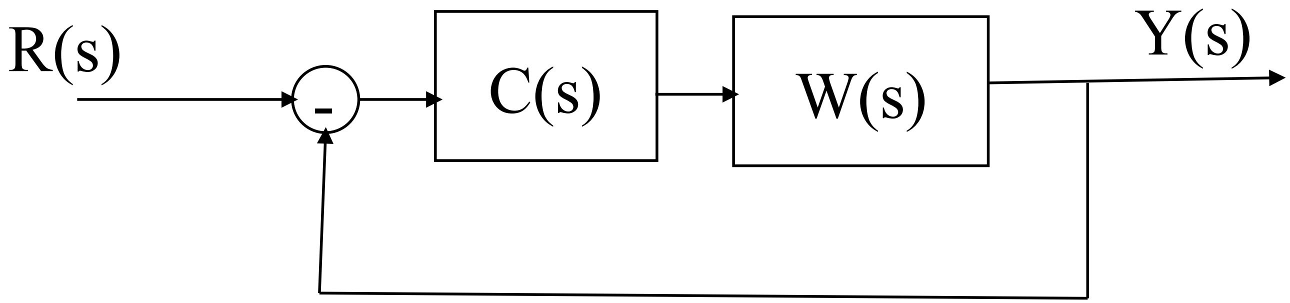
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{\ell\times n}$$
$$A, B, C = \text{const}$$

$$W(s) = \frac{C \text{adj}(sI - A)^T B}{\det(sI - A)}$$
$$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n = 0$$

the characteristic polynomial of *A*, the denominator of *W(s)*

Nyquist stability criterion

- Nyquist stability criterion provides information about stability of the closed-loop system



Open-loop transfer function: C(s)W(s)

$$\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)W(s)}{1 + C(s)W(s)}}$$

Stability: The zeros of [1+C(s)W(s)] must be outside the stability region

The poles of C(s)W(s)=D(s) are the same as the poles of 1+C(s)W(s)

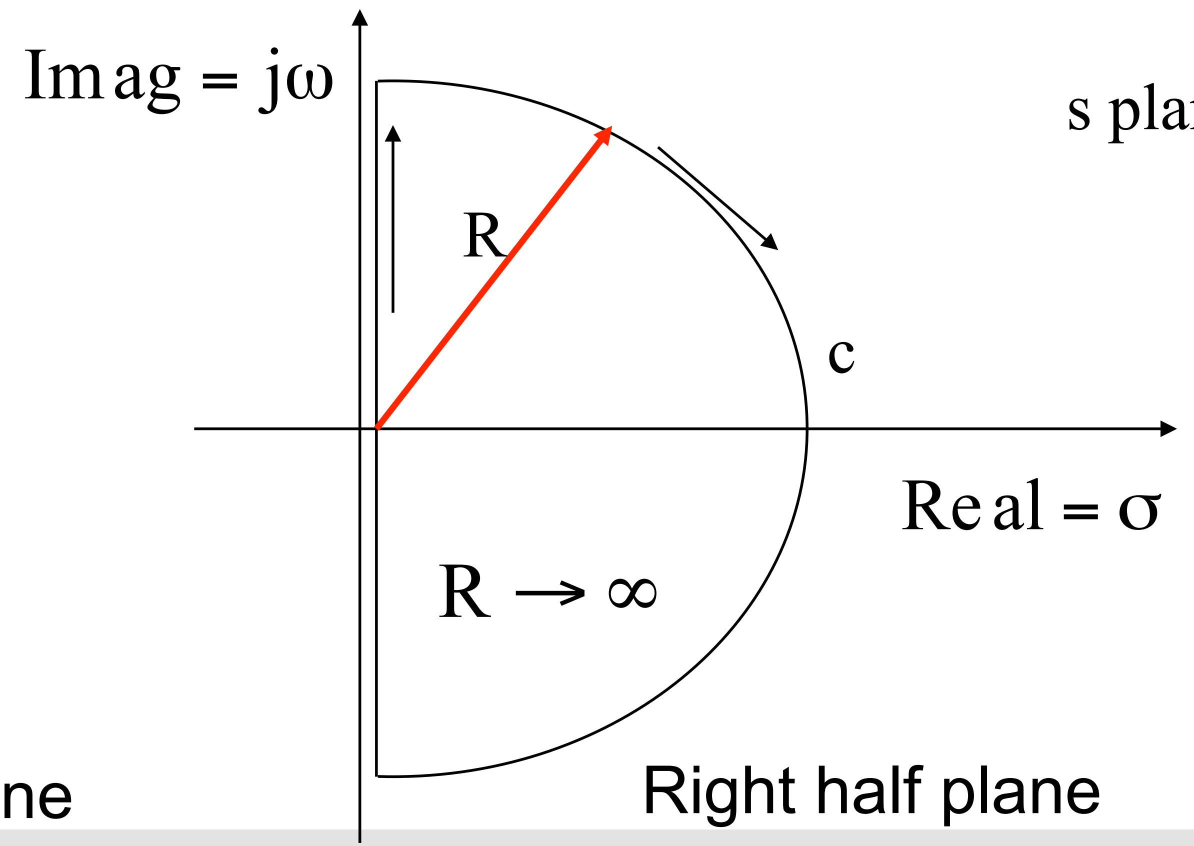
Cauchy's principle of argument

- The Nyquist stability criterion can be understood by using the argument principle (Cauchy's principle of argument).

- Cauchy's principle of argument: For a function $F(s)$, analytical except in a finite number of poles. If the function $F(s)$ has Z zeros and P poles in an area Γ_s then Z-P= number of times the curve $F(s)$ encircles the origin when s follows the boundary Γ_s in positive direction (counter clockwise)

Nyquist contour

- The Nyquist contouris a polar plot of the function *D*(*s*)=1+C(s)W(s) when *s* travels around the contour (or we ‘sweep’ the frequencies):



Left half plane

Right half plane

Nyquist stability criterion

- The number of unstable closed-loop poles (Z) is equal to the number of unstable open loop poles (P) plus the number of encirclements of the origin (N) of the Nyquist contour of the complex function $D(s)$

$$Z=N+P$$

•N is positive for encirclements in direction clockwise
•N is negative for encirclements in opposite direction clockwise

- The Nyquist criterion uses this transformation:

$$D'(s)=D(s)-1=C(s)W(s)$$

- Then, the function C(s)W(s) is plotted for s following the contour and the encirclements of the Nyquist plot around the point [-1, j0] are counted.

Nyquist stability criterion

- If the system C(s)W(s) has no poles in the right half plane, the closed loop system $\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)W(s)}{1 + C(s)W(s)}}$

  is stable if and only if the Nyquist plot does not encircle the point (-1,0).

- If the system C(s)W(s) has poles in the right half plane, the closed loop system

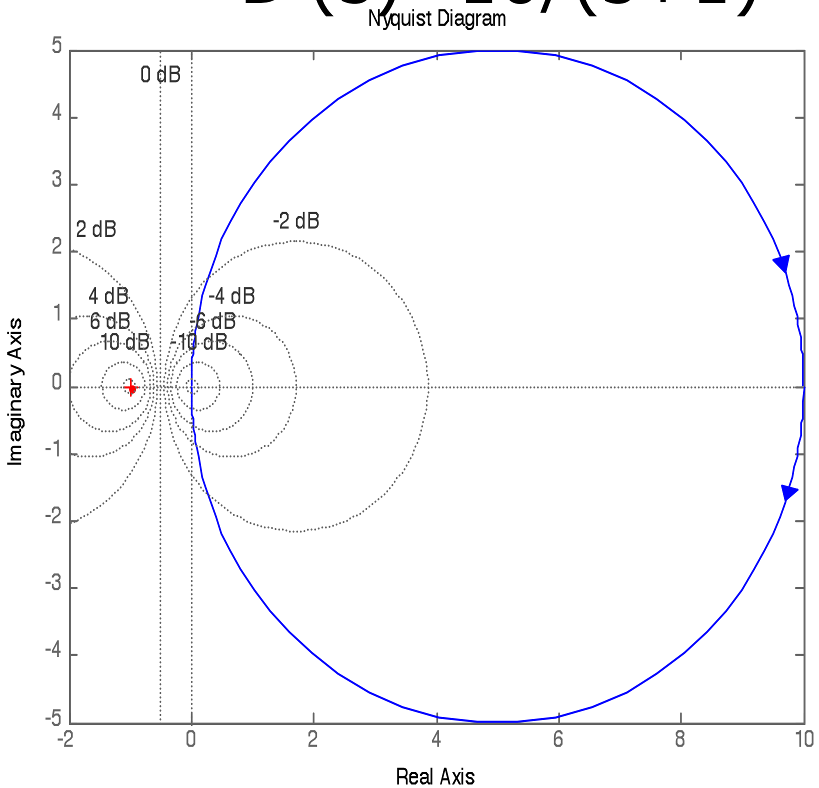
  $$\boxed{\frac{Y(s)}{R(s)} = \frac{C(s)W(s)}{1 + C(s)W(s)}}$$

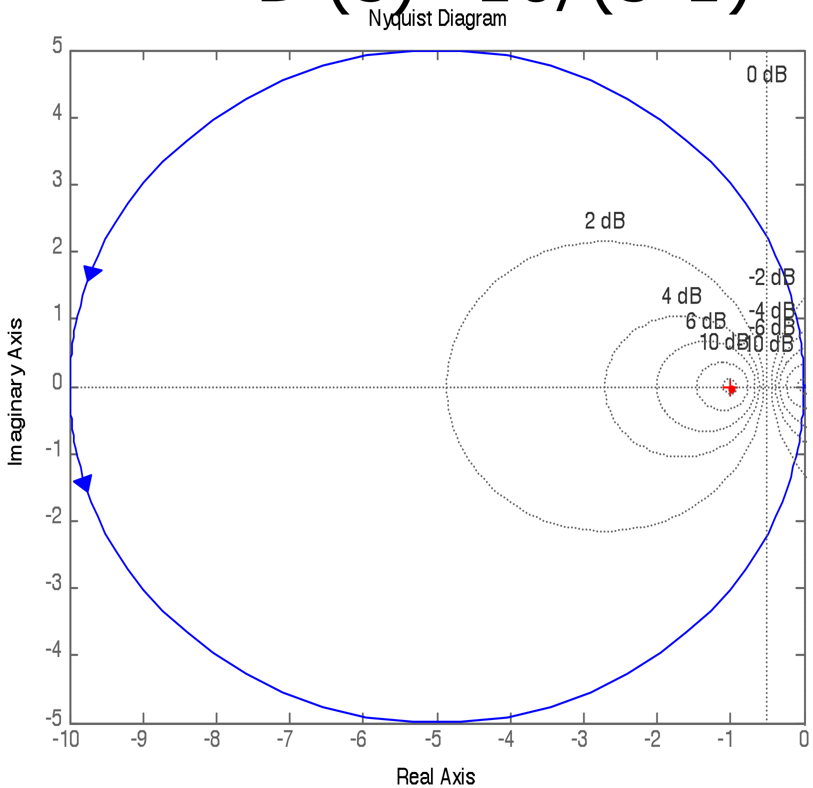
  is stable if and only if the number of encirclements of the point (-1,0) in clockwise direction of the Nyquist plot is equal to the number of poles in the right half plane.

Nyquist plot (Z=N+P)

- Nyquist diagrams are always symmetrical with respect to the real axis
- Two stable closed loop systems with open loop:

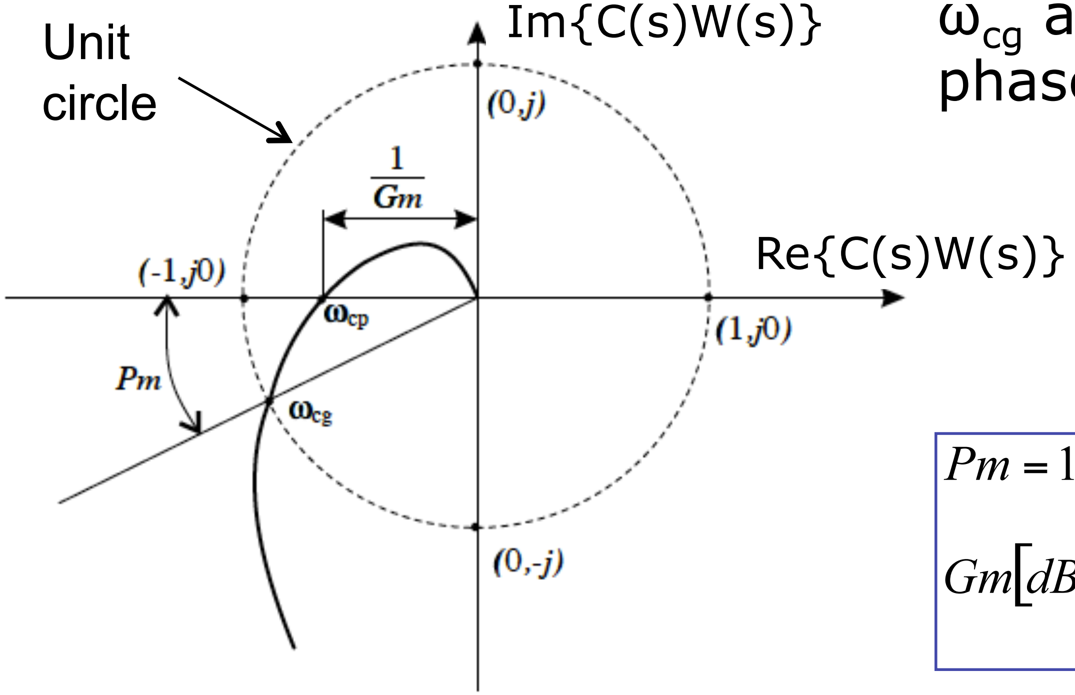
$D'(s)=10/(s+1)$ $D'(s)=10/(s-1)$





Gain and phase stability margins

- The margins give us information of how close the curves are to encircle the point (-1,0).



ω_{cg} and ω_{cp} are the gain and phase crossover frequencies

$$\left|C(j\omega_{cg})W(j\omega_{cg})\right| = 1$$

$$\arg\left\{C(j\omega_{cp})W(j\omega_{cp})\right\} = 180^{\circ}$$

$$Pm = 180^{\circ} + \arg\left\{C(j\omega_{cg})W(j\omega_{cg})\right\}$$

$$Gm[dB] = 20\log\frac{1}{\left|C(j\omega_{cp})W(j\omega_{cp})\right|}[dB]$$

Pm – phase stability margin
Gm – gain stability margin

Nonlinear systems versus linear

- Nonlinear system in state-space form

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), u(t)\right) \\ y(t) &= h\left(x(t), u(t)\right) \end{aligned}$$

- Linear system in state-space form

$$\begin{aligned} f\left(x(t), u(t)\right) &= Ax(t) + Bu(t) \\ h\left(x(t)\right) &= Cx(t) + Du(t) \end{aligned}$$

- Nonlinear systems versus linear
  - Superposition principle is not valid for nonlinear systems
  - The principle of frequency preservation does not apply to nonlinear systems
  - Linear models can be seen as approximations of nonlinear systems → linearization

Stability for equilibria via linearization

- A nonlinear system in state-space form

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$

- Can be described, by using linearization, in the vicinity of the equilibrium (x_0,u_0) by the linear system

$$\dot{z}(t) = Az(t) + Bv(t)$$

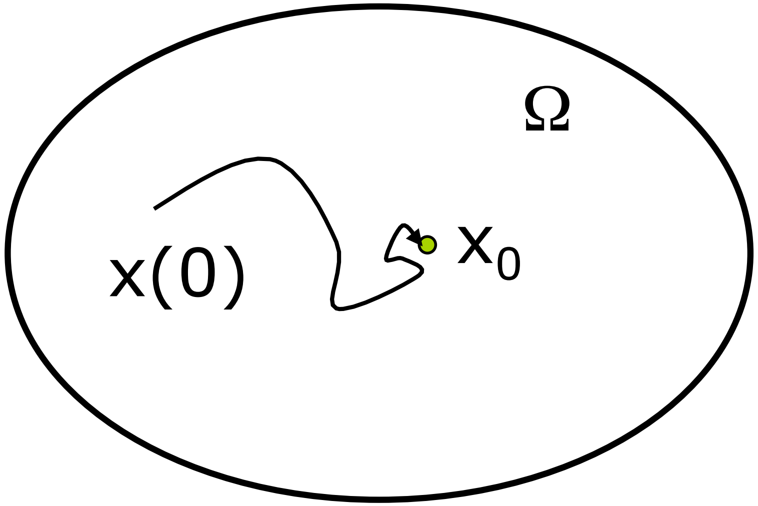
where $z=x-x_0$ and $v=u-u_0$.

$$a_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{(x,u)=(x_0,u_0)} \qquad b_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{(x,u)=(x_0,u_0)}$$

- If all eigenvalues of A have strictly negative real part, then (x_0,u_0) is an asymptotically stable equilibrium.
- If any of the eigenvalues of A has strictly positive real part then (x_0,u_0) is an unstable equilibrium
- If none of the eigenvalues of A has positive real part but there are eigenvalues on the imaginary axis then the equilibrium can be either stable or unstable.

Stability of equilibria via linearization

- To establish stability properties of equilibria there is no need in studying the solutions of the nonlinear system in question. Stability properties are defined by the stability properties of the linearized system and linear theory is sufficient.
- If an equilibrium is asymptotically stable then it is surrounded by an attraction domain. All solutions that start within the attraction domain converge to the equilibrium. The size of an attraction domain is typically difficult to estimate.



$$x(0) \in \Omega \Rightarrow x(t) \rightarrow x_0, t \rightarrow \infty$$

Summary

- There are many notions of stability.
- Stability is a system property in LTI systems and defined by the eigenvalues of the system matrix in state space form or the denominator of the transfer function, i.e. system poles.
- The Nyquist stability criterion provides useful information about stability of closed-loop LTI systems.
- Nonlinear systems are much more difficult to analyze than linear ones.
- Linearization of nonlinear systems can be used to investigate stability of equilibria. In some cases though, it is necessary to analyze the nonlinear system, anyway.