Lecture 8: Pole placement and observers

Outline

- Pole placement in state-space form
- Observers
- Feedback from reconstructed states

State feedback controller

- Continuous time LTI

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- Discrete time LTI

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

State feedback: $u(t) = -Kx(t) + K_r r(t)$

Closed-loop continuous time

$$\dot{x}(t) = (A - BK)x(t) + BK_r r(t)$$

Closed-loop discrete time

$$x(t+1) = (A - BK)x(t) + BK_r r(t)$$

- The aim of control is to drive the output *y* to a given reference value *r*.
- Controller design:
  - Select *Kr* so that the steady state output to be equal to the reference value *r*.
  - Select *K* so that *A-BK* corresponds to an asymptotically stable system.
- Stability *of the* closed-loop system is defined by the eigenvalues of *A-BK*
- *Kr* does not affect the stability of the closed-loop system but does affect the steady state values.

Pole placement by state feedback

- The feedback gain *K* can be selected so that *A-BK* attains arbitrary pre-defined eigenvalues if and only if the controllability matrix has full rank

$$\operatorname{rank} \mathcal{S}(A,B) = n$$

$$\mathcal{S}(A,B) = \begin{bmatrix} B & AB & \ldots A^{n-1}B \end{bmatrix}$$

- The eigenvalues of *A-BK* are the poles of the closed-loop system, i.e. the roots of the characteristic polynomial

$$\alpha_c(A-BK) = \det(sI - A + BK) = 0$$

- The design procedure that places the eigenvalues of *A-BK* at a set of desired locations is called pole placement
  - Ackermann's formula is used for SISO systems

$$K = \begin{bmatrix} 0 & 0 & \ldots 1 \end{bmatrix} \mathcal{S}(A,B)^{-1} \alpha_c(A)$$

  - More advanced algorithms for MIMO system
- If the pair *(A,B)* is in controllability canonical form, then the pair *(A-BK,B)* is also in controllability canonical form.

State observers

- The state feedback controller assumes the state vector to be available for measurement.
- The state vector is composed of internal variables that are not necessary measured and their estimates are therefore sought.
- System model is available and can be used for estimation of non-measurable internal variables
- Continuous time:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- Discrete time:

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

$$\dot{\hat{x}} = A\hat{x} + Bu; \;\; \hat{x}(0) = \hat{x}_0; \;\; \hat{x} - \text{state estimate}$$

- With known *A* and *B*, a mathematical model can be fed with the same control signal as the system itself

State observers, contd.

- State estimation error for the model

$$\dot{\hat{x}} = A\hat{x} + Bu$$

$$e(t) = x(t) - \hat{x}(t)$$

$$\dot{e} = Ae$$

$$e(t) = \exp(At)(x_0 - \hat{x}_0)$$

- When *A* has all the eigenvalues in the left half-plane

$$e(t) \to 0, t \to \infty$$

- What to do when *A* does not give fast enough convergence or unstable?
- Use feedback! *e(t)* cannot be measured but *Ce(t)* can be.

$$Ce(t) = Cx(t) - C\hat{x}(t) = y - \hat{y};\ \ \hat{y} = C\hat{x}$$

$$\dot{e} = Ae - LCe$$

- Observer:

•Continuous time

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

•Discrete time

$$\hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right)$$

Design of state observers

- The feedback gain *L* is selected to make the state estimation error to go to zero asymptotically

$$\dot{e} = (A - LC)e \qquad\qquad e(t+1) = (A - LC)e(t)$$

- Design problem: choose *L* so that all eigenvalues of *A-LC* are within the stability domain.
- Note: any square matrix *A* has the same eigenvalues as A^T.

$$\det(sI - A + LC) = \det(sI - A^T + C^T L^T) = 0$$

- Controller: If *(A,B)* is a controllable pair then the eigenvalues of *A-BK* can be assigned arbitrarily by the choice of *K*.
- Observer: If *(A,C)* is an observable pair then the eigenvalues of *A-LC* can be assigned arbitrarily by the choice of *L*.
- If *(A,B)* is an controllable pair then (A^T,B^T) is an observable pair.
- Observer design can therefore be reduced to controller design by pole placement in A^T-C^TL^T that gives the value of *L*.

Feedback from reconstructed states

- Plant in state space form:

$$\dot{x} = Ax + Bu \qquad x(t+1) = Ax(t) + Bu(t)$$
$$y = Cx \qquad y(t) = Cx(t)$$

- Observer to estimate the states from the output

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \qquad \hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right)$$

- Feedback stabilizing controller from estimated states:

$$u(t) = -K\hat{x}(t)$$

- Feedback stabilizing controller with reference signal

$$u(t) = -Kx(t) + K_r r(t)$$

- The closed-loop system:

$$\dot{x} = Ax + Bu \qquad x(t+1) = Ax(t) + Bu(t)$$
$$\dot{e} = (A - LC)e \qquad e(t+1) = (A - LC)e(t)$$

Feedback from reconstructed states, contd

- Closed-loop system in state-space form

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} x(t) + \begin{bmatrix} BK_r \\ 0 \end{bmatrix} r(t) \\ y(t) &= \begin{bmatrix} C & 0 \end{bmatrix} x(t) \end{aligned}$$

- Transfer function *Y(s)=W(s)R(s)* $W(s) = C(sI - A + BK)^{-1} BK_r$
- The state estimation error is not controllable but observable
- Characteristic polynomial of the closed-loop system

$$\det(sI - A + BK)\det(sI - A + LC) = 0$$

- The closed-loop system is stable whenever the following systems are stable:

$$\dot{e} = (A - LC)e \qquad \dot{x} = (A - BK)x$$

- The dynamics of the observer should be faster than the dynamics of the state feedback controller to alleviate the impact of the state estimation error on the controller performance

Summary

- For a controllable LTI system with measurable state vector its dynamics can be arbitrarily changed by a static state feedback.
- For an observable system, all the state variables can be estimated from measurements of the input and the output by an observer.
- For an observable and controllable system, its dynamics can be arbitrarily changed by feeding back the estimates of the state variables provided by a state observer.
- The observer and controller design in an observer-based controller should be performed so that the observer dynamics is much faster than the controller dynamics.