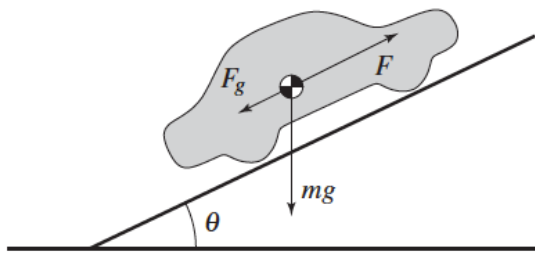# Lecture 9: Sensitivity and robustness

Outline

- Sensitivity
- Robustness

Example: Cruise control - The process

- The control goal is to maintain constant velocity in the presence of disturbances and model uncertainties (parameter variations).

- Disturbances can appear from many different sources:
  - Changes in the slope of a road
  - Aerodynamic forces
  - Rolling resistance

- A detailed model can be very complicated, and the design of the controller can a hard task!

$$m\frac{dv}{dt} = \alpha_n u T(\alpha_n v) - mg C_r \operatorname{sgn}(v) - \frac{1}{2}\rho C_v A v^2 - mg\sin\theta$$

Engine force   Rolling friction   Aerodynamic drag   Gravitational disturbance force

Example: Cruise control - The model

- For this example, the controlled variable is the velocity (v), the manipulated variable is the throttle (u) (throttle→ torque → force for moving the car), and the disturbance is the slope angle (θ).

- By using linearization around an equilibrium point (v_e, u_e, θ_e), the following linear model can be used for controller design, e.g. by using pole placement.

$$\frac{dV(t)}{dt} = aV(t) + bU(t) - b_g\Theta(t)$$

$$a = \frac{u_e a_n^2 T'(a_n v_e) - \rho C_v A v_e}{m}$$
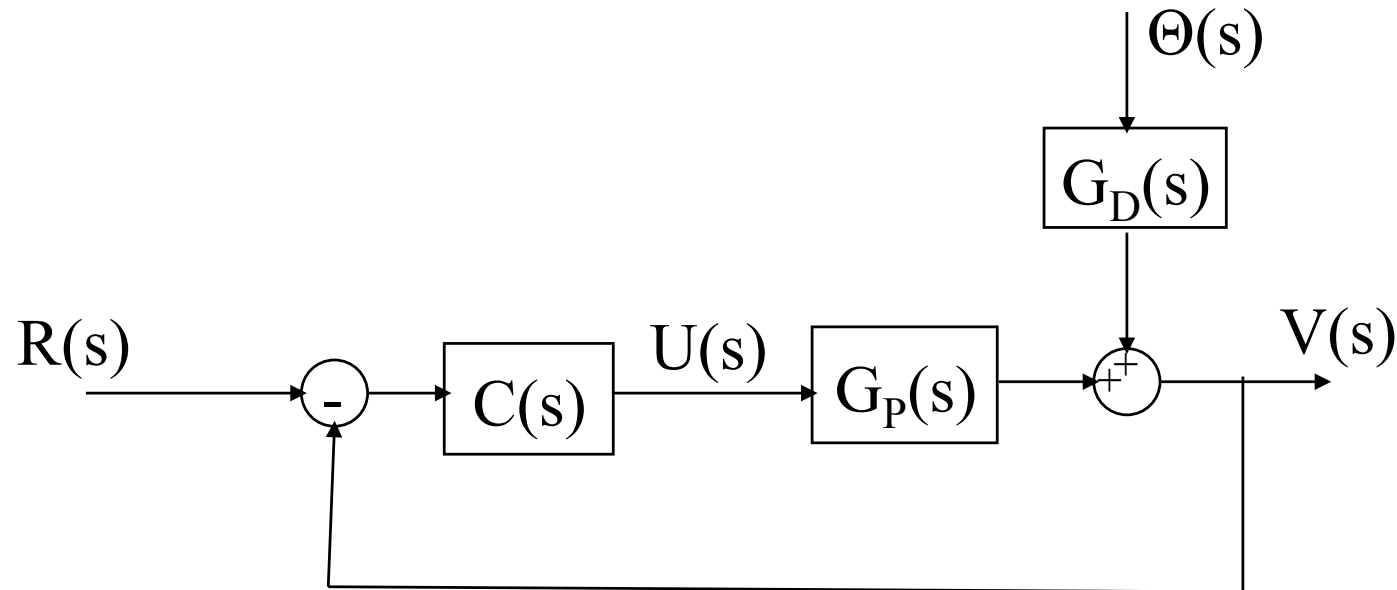
$$b = \frac{a_n T(a_n v_e)}{m}$$

$$b_g = g\cos\theta_e$$

- The transfer function for this system is

$$V(s) = G_p(s)U(s) + G_d(s)\Theta(s)$$

Example: Cruise control - The controller

$$V(s) = \frac{G_p(s)C(s)}{1 + G_p(s)C(s)} R(s) + \frac{G_D(s)}{1 + G_p(s)C(s)} \Theta(s)$$
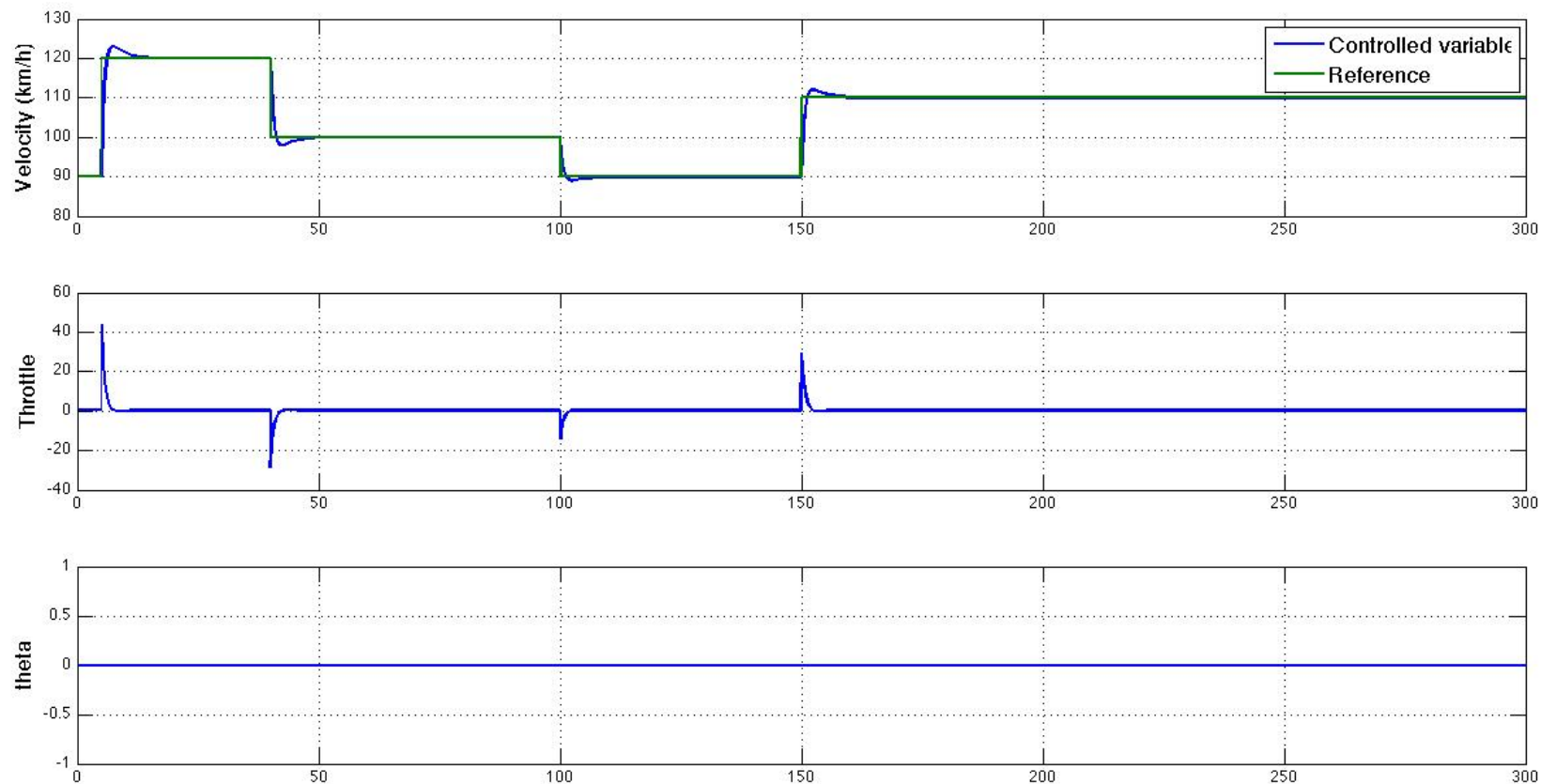
$$C(s) = K_P + \frac{K_I}{s}$$ PI controller

- The PI design by pole placement must satisfy that the desired characteristic polynomial $\alpha(s) = 1 + G_p(s)C(s)$
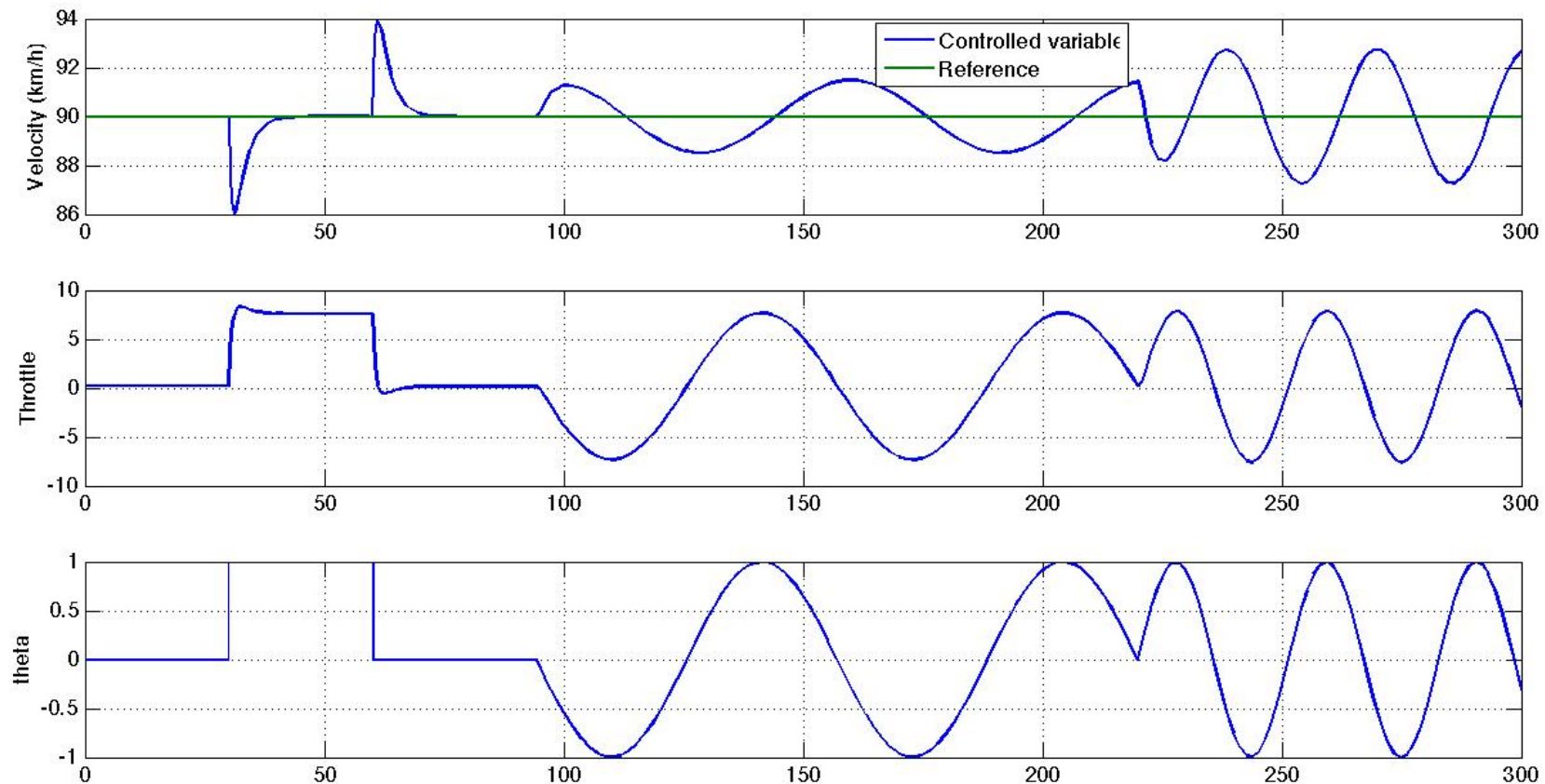
Example: Cruise control using a PI controller – The nominal performance

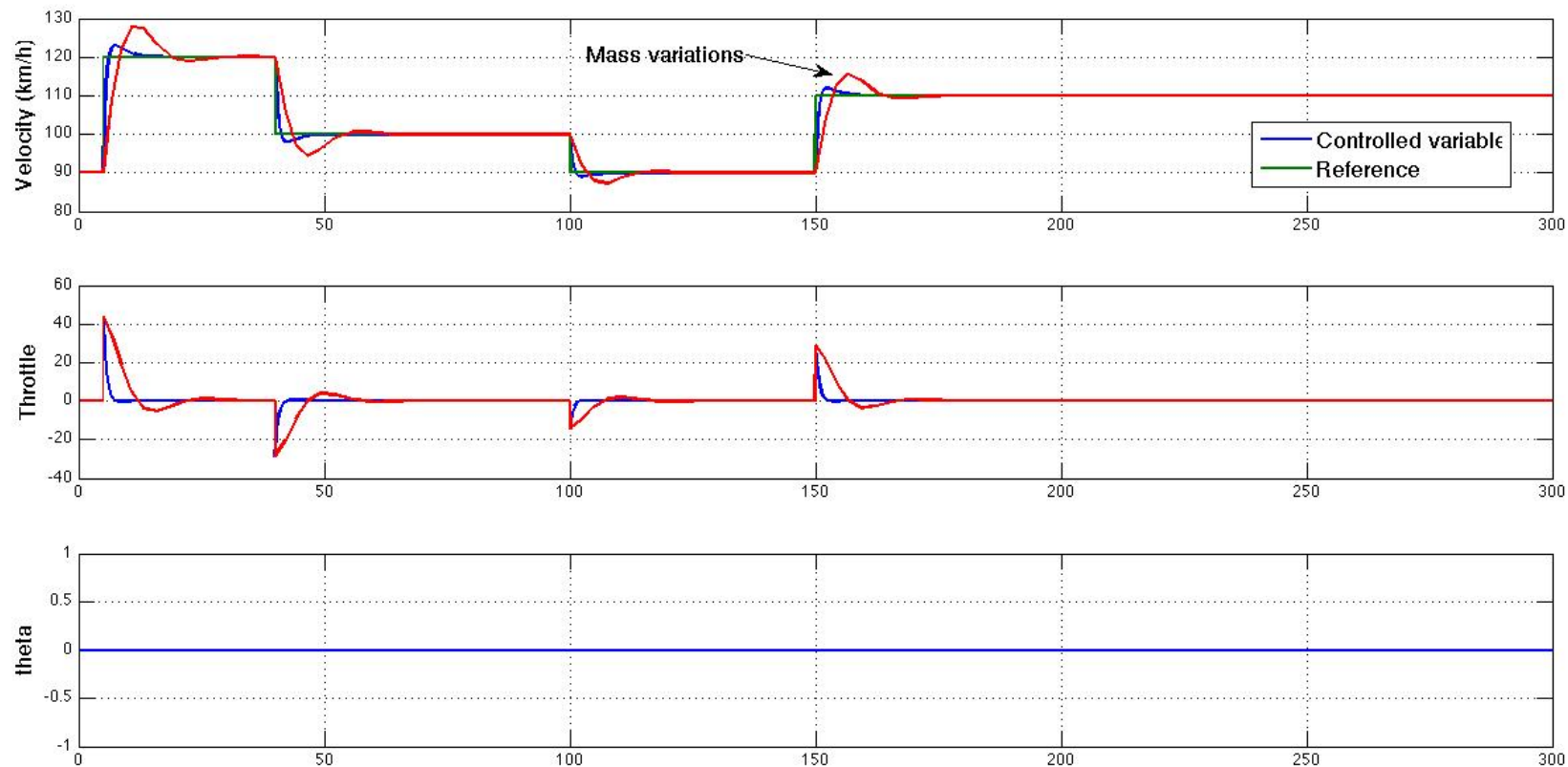- Changes on the velocity reference value

Example: Cruise control using a PI controller - Disturbances

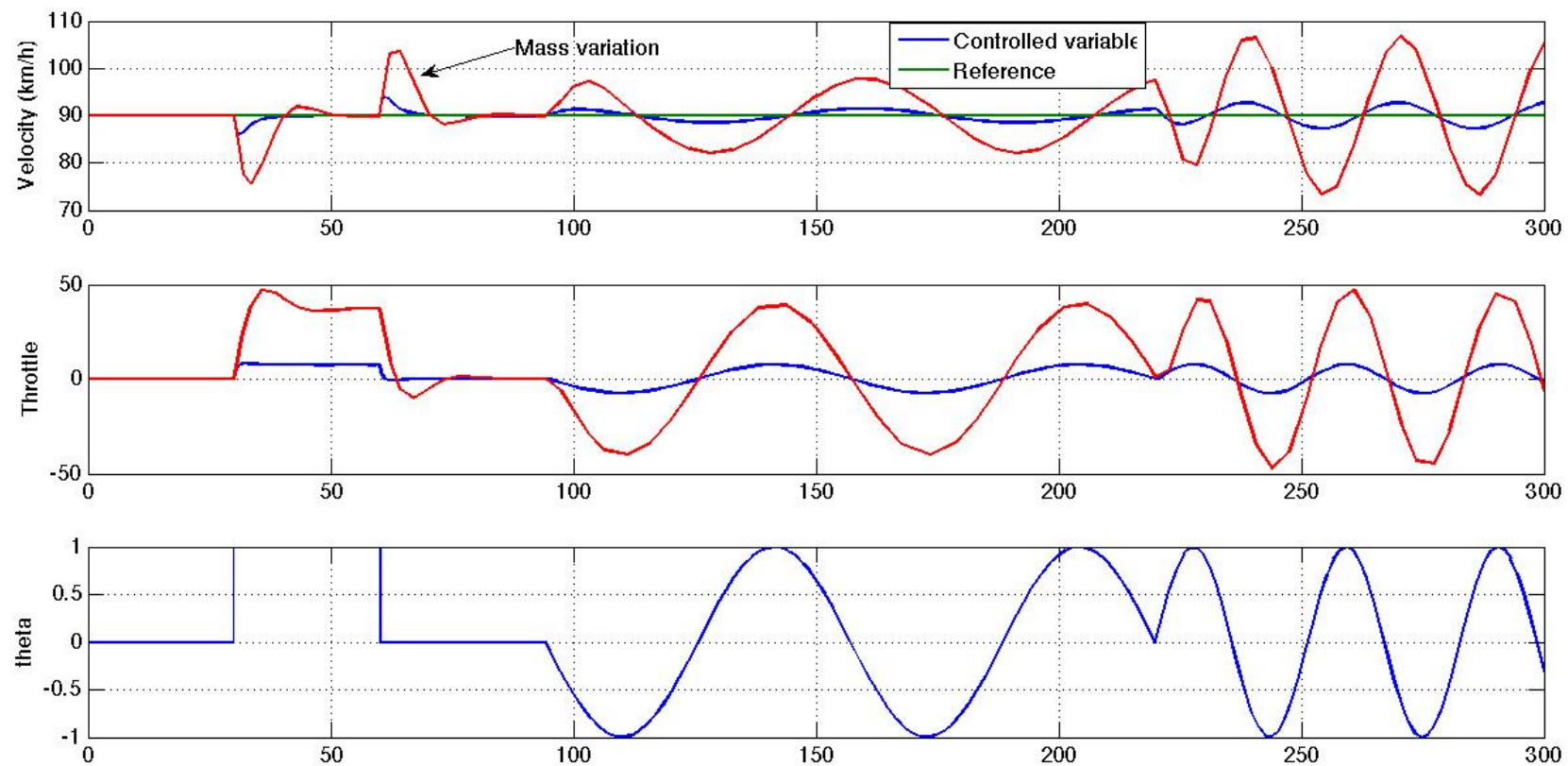- Slope changes (disturbance) with constant reference

Example: Cruise control using a PI controller - Parameter variations

- Mass variations (parameter) and reference changes

Example: Cruise control using a PI controller - Parameter variations

- Mass variation (parameter) and slope changes (disturbances)

Some remarks

- We have designed feedback controller in order that the closed loop system is stable and the output tracks the reference signal.

- The closed-loop control system is affected by the disturbances as well as parameter variations.

- It is desirable that the closed-loop system is insensitive to disturbances and parameter variations.

- It is desirable that the closed-loop system remains stable despite of parameter variations.

Sensitivity function
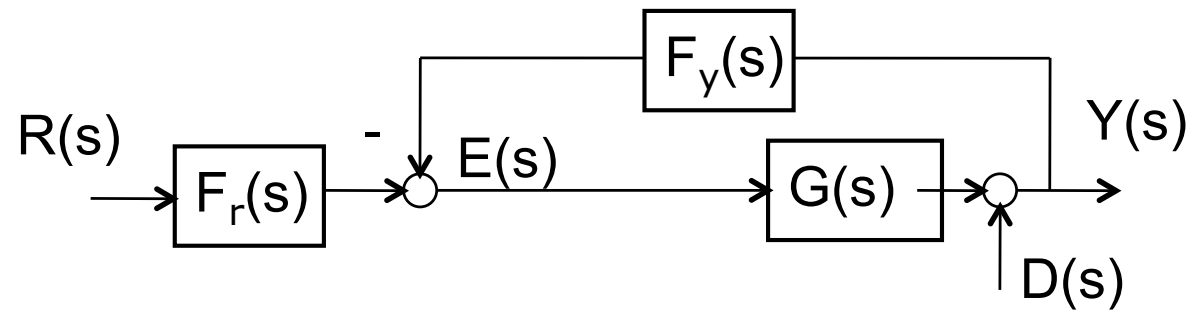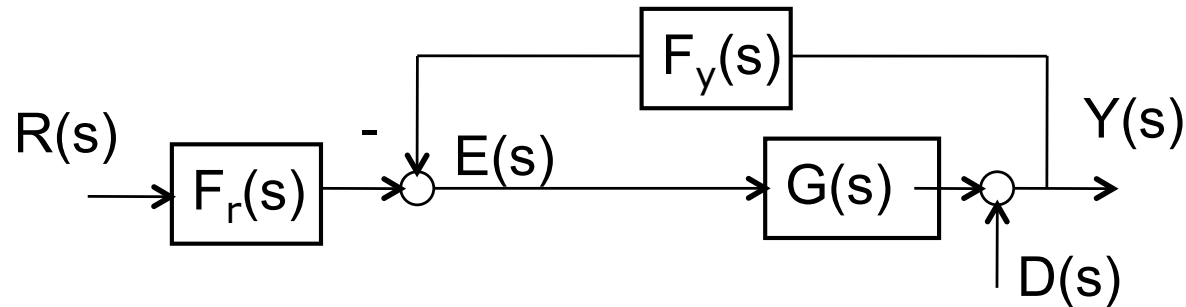
Sensitivity function

System equations:

$$E(s) = F_r(s)R(s) - F_y(s)Y(s)$$

$$Y(s) = G(s)E(s) + D(s)$$

Closed-loop system (SISO) with disturbance:

$$Y(s) = \frac{G(s)F_r(s)}{1 + G(s)F_y(s)}R(s) + \frac{1}{1 + G(s)F_y(s)}D(s)$$

$$W(s) = \frac{G(s)F_r(s)}{1 + G(s)F_y(s)}$$ The closed-loop transfer function

$$S(s) = \frac{1}{1 + G(s)F_y(s)}$$ The sensitivity function

Sensitivity function - Disturbances

- $F_r(s)$ and $F_y(s)$ have to yield stability of the closed loop system and fulfill the design objective

- Design objectives:
  - Disturbance rejection
  - Reference tracking

- Disturbances with frequencies such that $|S(j\omega)|<1$ are attenuated by closed-loop system, but disturbances with frequencies such that $|S(j\omega)|>1$ are amplified.

Sensitivity function, contd.

- If $F_r(s)=F_y(s)=C(s)$,
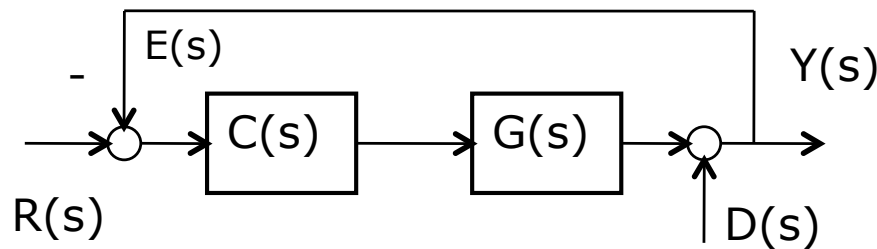
- Then the output of the closed-loop system is

$$Y(s) = \frac{C(s)G(s)}{1+C(s)G(s)}R(s) + \frac{1}{1+C(s)G(s)}D(s)$$

- **Disturbance rejection**: To minimize the impact of $d(t)$ on $y(t)$, $C(jw)$ should be large in the frequency range of $d(t)$, (where $D(jw)$ is large)

- **Reference tracking**: To make y(t) follow r(t), $C(jw)$ should be large in the frequency range of r(t), (where $R(jw)$ is large)

- Stability problems usually arise for high gain design

Sensitivity function – Model errors

• Sensitivity function also provides information about how the parameter variations are affecting the closed-loop output.

R(s) → $F_r(s)$ → (−) $E_0(s)$ → G(s) → $Y_0(s)$

$F_y(s)$

$\Delta_G(s)G(s)$

$G_0 = (1 + \Delta_G)G$
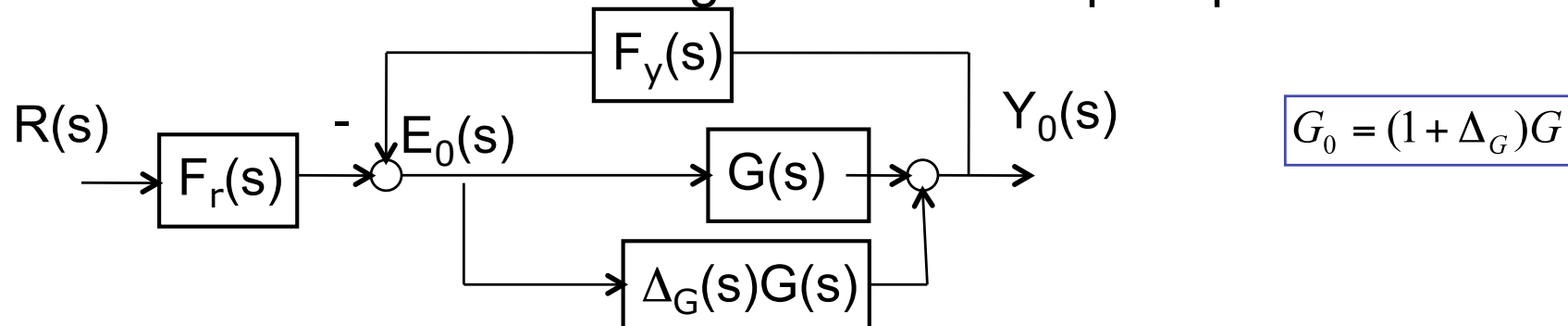
•G is the nominal model, G_0 is the true model and Δ_G is the relative model error

•The closed-loop output for the true model is related to the nominal closed-loop output by

$$Y_0 = (1 + \Delta_Y)Y$$
$$\Delta_Y = S_0\Delta_G$$
$$S_0 = \frac{1}{1 + G_0 F_y}$$

• S_0 describes how the relative model error Δ_G is transformed to a relative output error Δ_Y

•The sensitivity function should also kept low (by controller design) in the frequency range where the plant model is uncertain.

Robustness

- Robustness is a property of feedback.
- A simple robustness criterion is that the Nyquist curve is sufficiently far from the critical point (-1,0).
- The model errors that are allowed without endangering the stability of the closed loop system can be measure by the robustness of the closed-loop system.
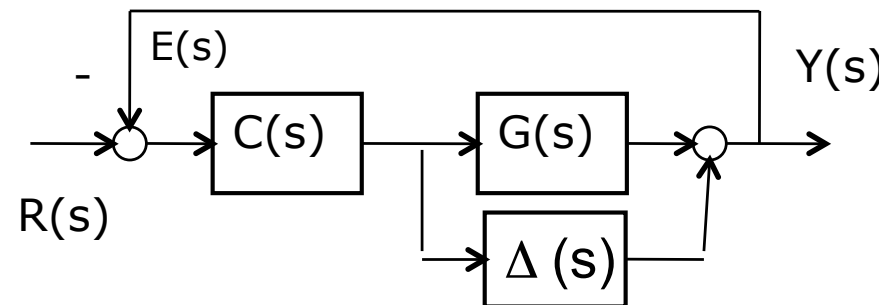
Δ (s) -The model errors

- The open loop transfer function changes from [C(s)G(s)] (the nominal case) to [C(s)G(s) +C(s)Δ (s)] (the true case)
- The system remains stable if the model errors Δ (s) are bounded so that the 'true' Nyquist curve doesn't encircle the point (-1,0) (assuming that G(s) and Δ (s) do not have zeros on the right half plane)

$$\left|C(j\omega)\Delta(j\omega)\right| < \left|1 + C(j\omega)G(j\omega)\right|$$

$$\left|\Delta(j\omega)\right| < \left|\frac{1 + C(j\omega)G(j\omega)}{C(j\omega)}\right|$$ for all $\omega > 0$.

Summary

- Sensitivity functions give us information about how the output of the closed loop system is affected by disturbances and model errors.

- Robustness give us information about what model errors can be allowed in order that the closed loop system remains stable.

- A good controller design must provide to the closed loop system:

  - Small sensitivity to expected disturbances and model errors (in order to have little influence on the output)

  - Robustness against model uncertainties

- Robust closed loop system allow us the use of simplified models for controller design.

Lecture 10: Practical aspects/ Repetition

Outline

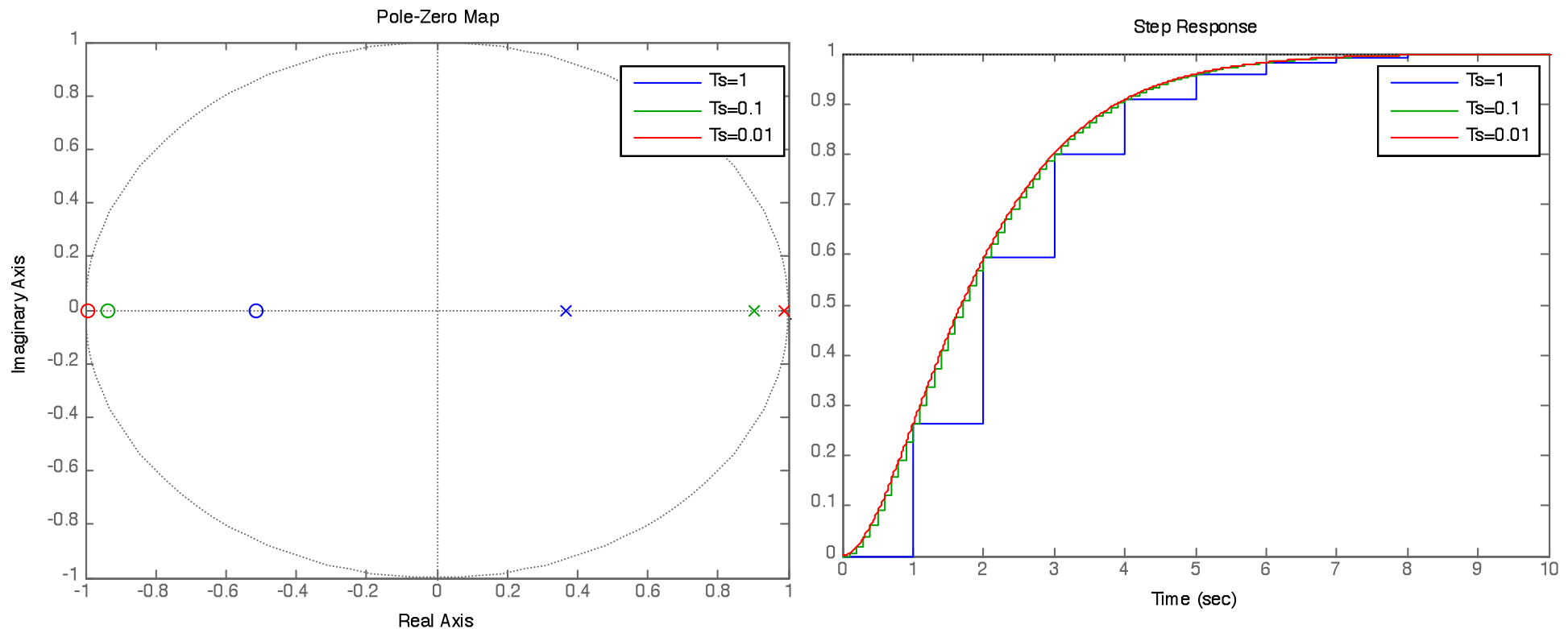- Sampling time
- Sampled data control

Sampling time on pole location (z=e^{Ts})

- Effect of the sampling time on the location of <u>poles</u> in the z-map

$$s = \sigma + j\omega$$

$$z = e^{Ts} = e^{T(\sigma + j\omega)} = e^{T\sigma + Tj\omega}$$

$$z = e^{T\sigma} \angle T\omega = e^{T\sigma}(\cos T\omega + j\sin T\omega)$$
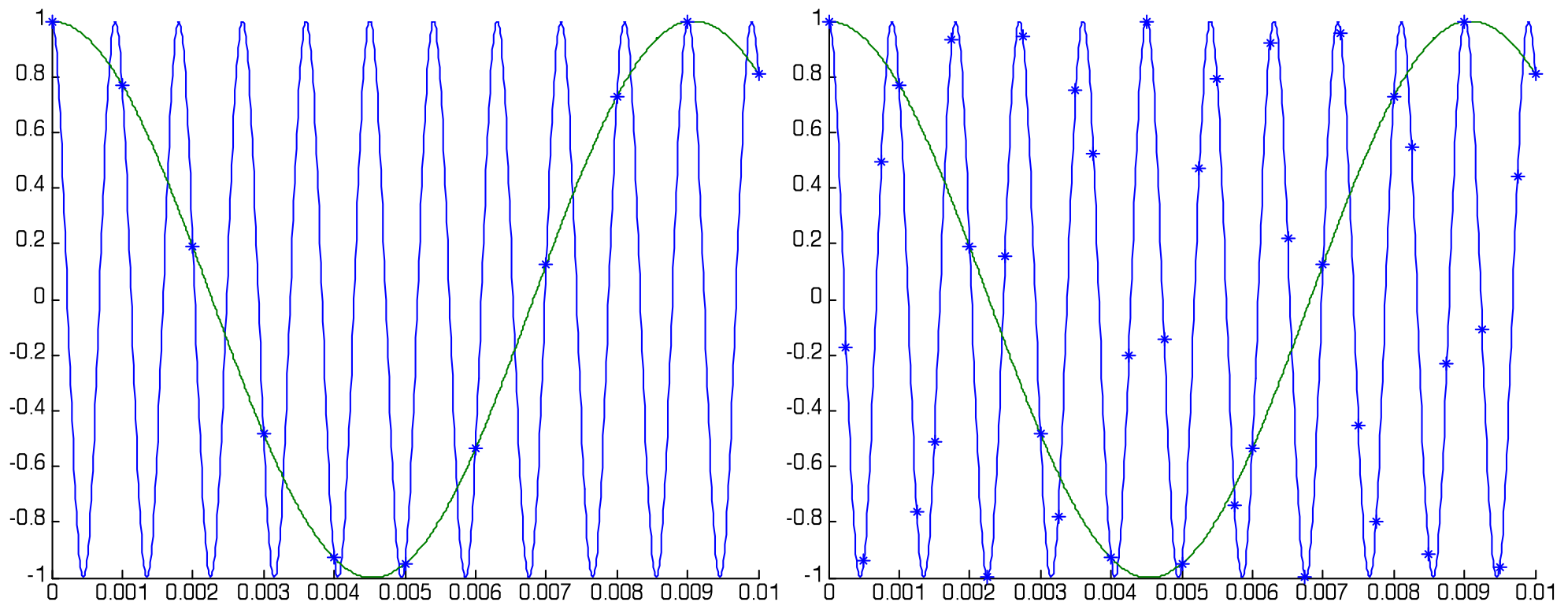
Sampling time

- Sampling time T_s – seconds; sampling frequency $w_c = 2\pi/T_s$ – rad/s
- **Nyquist-Shannon principle**: A <u>bandlimited</u> <u>analog signal</u> that contains no frequencies higher than *B* rad/s can be perfectly reconstructed from an infinite sequence of samples if the sampling time is less or equal π/B seconds.
- **Alias phenomenon**: a sampled continuous time sinusoid with frequency above $w_c = \pi/T_s$ (Nyqvist frequency) cannot be distinguished from a signal with frequency below w_c.
- **Anti-alias filter**: pre-sampling (continuous) low-pass filter minimizing higher frequency components
- **Thumb rule**: the sampling time is chosen as one tenth of the process time constant or less

$$W(s) = \frac{K}{Ts + 1}$$ *T* – time constant (s)
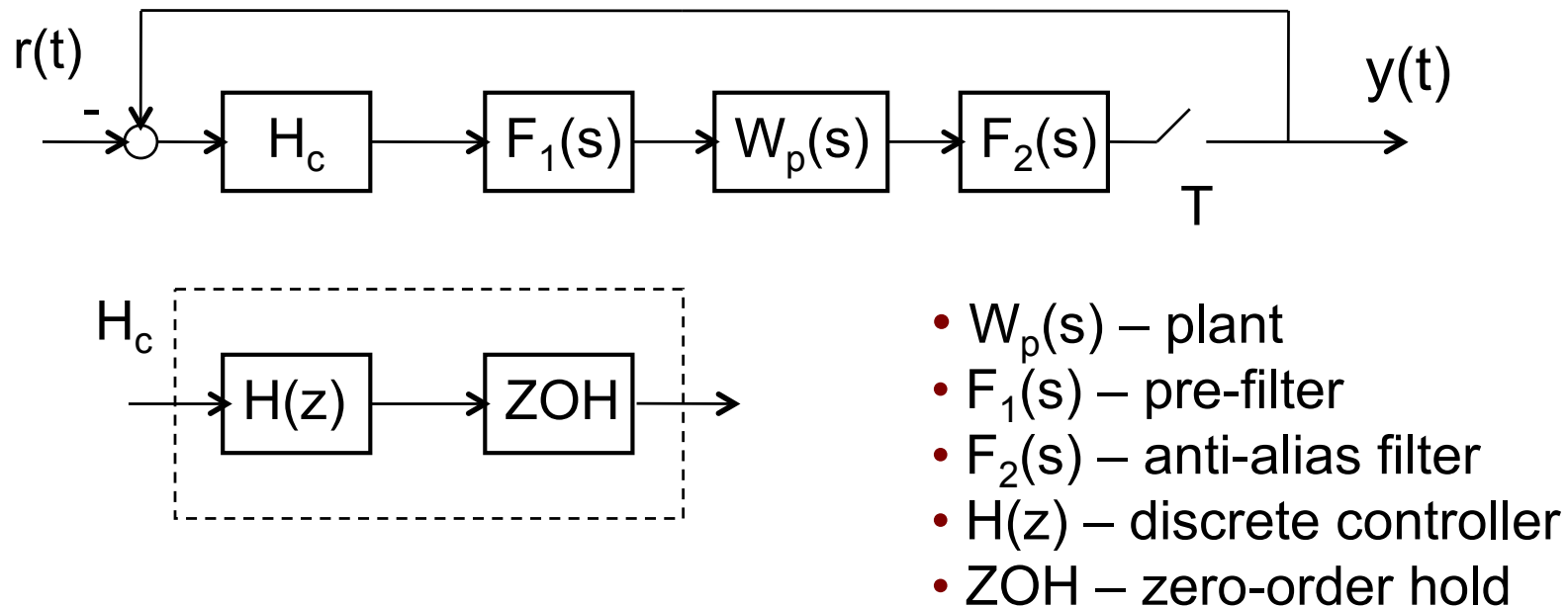
Aliasing example

- $y_1 = \cos(2\pi f_1 t)$, $f_1 = 110$ Hz
- $y_2 = \cos(2\pi f_2 t)$, $f_2 = 1110$ Hz
- $f_s = 1000$ Hz $f_s = 4000$ Hz

Sampled data control

Feedback sampled-data controller

- $W_p(s)$ – plant
- $F_1(s)$ – pre-filter
- $F_2(s)$ – anti-alias filter
- $H(z)$ – discrete controller
- ZOH – zero-order hold

- The plant and the filters are continuous
- The controller $H(z)$ is discrete
- Sampled-data design includes the controller and the filters

Summary

- Sampling time in a control system has to follow Nyqvist-Shannon principle

- Sampling time is typically selected from an experiment or a process model

- Discrete controllers for continuous plants are most suitable to design via sampled-data theory