Introduction to Computer Control Systems

Lecture 2: Representation of linear systems

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Today’s lecture: What and why?

Terminology

**Why:** Establish a framework and terminology to describe systems
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Terminology

Why: Establish a framework and terminology to describe systems

Mathematical representations of a system

Why: Enable analysis of system properties and design controllers
System example: Tank

On the board: a tank model

\[ u \]

\[ x \]

\[ W \]

\[ y \]
**Terminology**

- **Constants**: model quantities that do not change with time
  - **System parameters**: constants pertaining to system description
  - **Design parameters**: constants that can be selected to give the systems desired properties
- **Variables/signals**: model quantities that vary with time
**Terminology**

- **Constants**: model quantities that do not change with time
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- **Variables/signals**: model quantities that vary with time

**On the board**: exemplify!
Mathematical representations of LTI systems

\[ y(t) = G u(t) \]

**Figure:** Graphical representation of linear time-invariant LTI system \( G \).

Different forms of representation:
- (Input-output equations)
- State-space equations
- Transfer function
- Impulse response

**Q: Why?**

**A:** Using mathematical descriptions of how \( u(t) \) affects \( y(t) \) we may design rules for automatic computer control of \( u(t) \) to yield a desired output \( y(t) \approx r(t) \).
Mathematical representations of LTI systems

On the board:
- Rate of change in volume $dV(t)/dt = u(t) - y(t)$
- Model output flow proportional to height, $y(t) = \alpha x(t)$.

Figure: System $G$ with input $u(t)$ and output $y(t)$.
Number 0 Input-output equations

\[ y(t) = G(u(t)) \]

**Standard form:**

\[
y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} \dot{u} + b_n u,
\]

where \( y^{(n)} \triangleq \frac{d^n y(t)}{dt^n} \) is shorthand for \( n \)th derivative.
#0 Input-output equations

![Block Diagram]

Standard form:

\[ y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} \dot{y} + a_n y \]

\[ = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1} \dot{u} + b_n u, \]

where \( y^{(n)} \triangleq \frac{d^n y(t)}{dt^n} \) is shorthand for \( n \)th derivative.

On the board:

\[ \dot{y} + \frac{\alpha}{W} y = \frac{\alpha}{W} u \]
\#1 State-space equations

Standard form:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\(n\) internal variables (states) and \(m\) input signals:

\[
\begin{align*}
x(t) &= \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \\
u(t) &= \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}
\end{align*}
\]
#1 State-space equations

![Block diagram](image)

**Standard form:**

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

\(n\) internal variables (states) and \(m\) input signals:

\[
x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}
\]

**On the board:**

\[
\dot{x} = -\frac{\alpha}{W}x + \frac{1}{W}u \\
y = \alpha x
\]
#2 Transfer function

Derivative as a linear operator $p$ such that $py(t) = \dot{y}(t)$.
#2 Transfer function

- Derivative as a linear operator $p$ such that $py(t) = \dot{y}(t)$.
- Input output eqs. can then be written as:

$$(p^n + a_1 p^{n-1} + \cdots + a_n)y = (b_0 p^n + b_1 p^{n-1} + \cdots + b_n)u$$
#2 Transfer function

- Derivative as a linear **operator** \( p \) such that \( p y(t) = \dot{y}(t) \).
- Input output eqs. can then be written as:

\[
(p^n + a_1 p^{n-1} + \cdots + a_n)y = (b_0 p^n + b_1 p^{n-1} + \cdots + b_n)u
\]

- Enables **compact** way to describe entire system.
Laplace transform of signal $y(t)$:

$$Y(s) = \mathcal{L}\{y(t)\} = \int_{t=0}^{\infty} y(t)e^{-st} dt$$

yields a different and mathematically useful representation of $y(t)$ as a function of complex variable $s$. 
Laplace transform of signal $y(t)$:

$$Y(s) = \mathcal{L}\{y(t)\} = \int_{t=0}^{\infty} y(t)e^{-st} \, dt$$

yields a different and mathematically useful representation of $y(t)$ as a function of complex variable $s$.

Calculating $Y(s) = \mathcal{L}\{y(t)\}$ and $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ by hand is done using tables + partial fractions.
Useful property: Laplace transform of derivative

\[ \mathcal{L}\{d^n y(t)/dt^n\} = s^n Y(s), \]

when \( d^n y(t)/dt^n = 0 \) at \( t = 0 \). (Similar to differential operator \( d^n y(t)/dt^n = p^n y(t) \)!)
Useful property: Laplace transform of derivative

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Input output eqs. can then be written as:

\[
(s^n + a_1 s^{n-1} + \cdots + a_n)Y(s) = (b_0 s^n + b_1 s^{n-1} \cdots + b_n)U(s)
\]
Useful property: Laplace transform of derivative

\[ L\{d^n y(t)/dt^n\} = s^n Y(s), \]

when \( d^n y(t)/dt^n = 0 \) at \( t = 0 \). (Similar to differential operator \( d^n y(t)/dt^n = p^n y(t) \)!) 

Input output eqs. can then be written as:

\[(s^n + a_1 s^{n-1} + \cdots + a_n)Y(s) = (b_0 s^n + b_1 s^{n-1} \cdots + b_n)U(s)\]

Standard form:

\[ Y(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} U(s) \]

Transfer function \( G(s) \) is a compact way to describe entire system
#2 Transfer function, cont’d

- **Useful property**: Laplace transform of derivative

  \[ \mathcal{L}\{d^n y(t)/dt^n\} = s^n Y(s), \]

  when \( d^n y(t)/dt^n = 0 \) at \( t = 0 \). (Similar to differential operator \( d^n y(t)/dt^n = p^n y(t) \)!

- Input output eqs. can then be written as:

  \[(s^n + a_1 s^{n-1} + \cdots + a_n)Y(s) = (b_0 s^n + b_1 s^{n-1} \cdots + b_n)U(s)\]

- **Standard form**:

  \[ Y(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \quad U(s) \]

  \[ G(s) \]

- **Transfer function** \( G(s) \) is a compact way to describe entire system

  **On the board**: \( G(s) = \frac{\alpha/W}{s + \alpha/W} \)
#3 Impulse response

Useful property: Multiplication $W(s)Z(s)$ in Laplace domain is equivalent to convolution in time domain,

\[
\mathcal{L}^{-1}\{W(s)Z(s)\} = \int_{\tau=0}^{t} w(\tau)z(t-\tau)d\tau
\]
Useful property: Multiplication $W(s)Z(s)$ in Laplace domain is equivalent to convolution in time domain,

$$\mathcal{L}^{-1}\{W(s)Z(s)\} = \int_{\tau=0}^{t} w(\tau)z(t-\tau)d\tau$$

Recall we had $Y(s) = G(s)U(s)$, then

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$
$$= \mathcal{L}^{-1}\{G(s)U(s)\}$$
$$= \int_{\tau=0}^{t} g(\tau)u(t-\tau)d\tau$$
#3 Impulse response

\[ u(t) \xrightarrow{G} y(t) \]

- **Useful property**: Multiplication \( W(s)Z(s) \) in Laplace domain is equivalent to convolution in time domain,

\[
\mathcal{L}^{-1}\{W(s)Z(s)\} = \int_{\tau=0}^{t} w(\tau)z(t - \tau) d\tau
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- Recall we had \( Y(s) = G(s)U(s) \), then

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\} = \int_{\tau=0}^{t} g(\tau)u(t - \tau) d\tau
\]

Thus output \( y(t) \) is entirely determined by a weighted combination of all past input signals \( u(t - \tau) \), since \( 0 < \tau \leq t \).
Given $G(s)$, the weights in $y(t) = \int_{\tau=0}^{t} g(\tau) u(t - \tau) d\tau$ can be computed as $g(\tau) = \mathcal{L}^{-1}\{G(s)\}$.
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Or estimated experimentally! On the board: Dirac delta.
Given $G(s)$, the weights in $y(t) = \int_{\tau=0}^{t} g(\tau)u(t - \tau)d\tau$ can be computed as $g(\tau) = \mathcal{L}^{-1}\{G(s)\}$.

Or estimated experimentally! On the board: Dirac delta. Suppose input is an impulse $u(t) = \delta(t)$. Then sifting property yields

$$y(t) = \int_{\tau=0}^{t} g(\tau)\delta(t - \tau)d\tau = g(t)!$$
Given $G(s)$, the weights in $y(t) = \int_{\tau=0}^{t} g(\tau)u(t - \tau) d\tau$ can be computed as $g(\tau) = \mathcal{L}^{-1}\{G(s)\}$.

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$$ y(t) = \int_{\tau=0}^{t} g(\tau)\delta(t - \tau) d\tau = g(t)! $$

Thus $g(\tau)$ equals the impulse response (output) of the system and contains all information to describe it.
Given \( G(s) \), the weights in \( y(t) = \int_{\tau=0}^{t} g(\tau)u(t-\tau)d\tau \) can be computed as \( g(\tau) = \mathcal{L}^{-1}\{G(s)\} \).

Or estimated experimentally! **On the board:** Dirac delta. Suppose input is an impulse \( u(t) = \delta(t) \). Then sifting property yields

\[
y(t) = \int_{\tau=0}^{t} g(\tau)\delta(t-\tau)d\tau = g(t)!
\]

Thus \( g(\tau) \) equals the impulse response (output) of the system and contains all information to describe it.

**On the board:** \( g(t) = \frac{\alpha}{W} e^{-\frac{\alpha}{W} t} \)
Representations of LTI systems: Overview

\[ G \]

\[ u(t) \rightarrow G \rightarrow y(t) \]

- **input-output eqs.**
- **state-space eqs.**
- **transfer function**
- **impulse response**
Representations of LTI systems: Model parameters

\[ y(t) = G(u(t)) \]

- Model parameters in:
  - Input-output eqs: \( a_1, \ldots, a_n \) and \( b_0, b_1, \ldots, b_n \)
  - State-space eqs: \( A, B, C \) and \( D \)
  - Transfer function: \( G(s) \)
  - Impulse response: \( g(\tau) \)

(and initial values) determine how system responds to input signal \( u(t) \)
Representations of LTI systems: Model parameters

$$y(t) = G(u(t))$$

Model parameters in

- input-output eqs: $$a_1, \ldots, a_n$$ and $$b_0, b_1, \ldots, b_n$$
- state-space eqs: $$A, B, C$$ and $$D$$
- transfer function: $$G(s)$$
- impulse response: $$g(\tau)$$

(and initial values) determine how system responds to input signal $$u(t)$$

*These quantities form mathematical tools for prediction and control!*
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