LTI system responses

**Why:** Characterize what your system does to a well-defined input signal. Control design criteria are often defined by using specifications of the step response.
Today’s lecture: What and why?

LTI system responses
Why: Characterize what your system does to a well-defined input signal. Control design criteria are often defined by using specifications of the step response.

Observability and controllability
Why: Is the system model such that we can observe all state changes through the output signal? Can we affect all the states using our input signal?
System example: Spring with input force

LTI system in state-space form:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

States and output:

\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]
System example: Spring with input force

Input $u(t)$ is an impulse $\delta(t)$

Temporal perspective
System example: Spring with input force

Input $u(t)$ is an impulse $\delta(t)$

State-space perspective $\mathbf{x}(t)$ at $t = 0^+$.
Input $u(t)$ is an impulse $\delta(t)$.

State-space perspective $x(t)$ at $t = 5$. 
System example: Spring with input force

Input $u(t)$ is an impulse $\delta(t)$

State-space perspective $\mathbf{x}(t)$ at $t = 20$. 
Input $u(t)$ is an impulse $\delta(t)$

State-space perspective $\mathbf{x}(t)$ at $t = 100$. 
LTI system response: transient and steady-state

LTI system in state-space form

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]
LTI system response: transient and steady-state

\[ y(t) = \int_{\tau=0}^{t} C e^{A(t-\tau)} B u(\tau) d\tau + D u(t) \]

Assume \( x(0) = 0 \) then

\[ y(t) = \left[ \text{input is a step } u(t) = 1 \text{ for } t \geq 0 \right] \]

\[ = CA^{-1} e^{At} B + -CA^{-1}B + D, \quad t \geq 0 \]

transient response  steady-state response
Input $u(t)$ is a unit step ($u(t) = 1$ for $t \geq 0$.)

Temporal perspective
LTI system response: transient and steady-state

Input $u(t)$ is a unit step ($u(t) = 1$ for $t \geq 0$.)

- Steady-state $y_{ss}$ (recall final value theorem) and overshoot $M$.
- Rise time $T_r$: time it takes for $y(t)$ to go from $0.1y_{ss}$ to $0.9y_{ss}$.
- Settling time $T_{sp}$: time it takes for $y(t)$ to stay within $(1 \pm p)y_{ss}$.
LTI system response: transient and steady-state

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Input $u(t)$ is a unit step ($u(t) = 1$ for $t \geq 0$.)

State-space perspective $\mathbf{x}(t)$ at $t = 100$. 
LTI system response: transient and steady-state

Input \( u(t) = \cos(\omega t) \) for \( t \geq 0 \).

Temporal: Note transient vs. stationary/steady-state of \( y(t) \).
LTI system response: transient and steady-state

Input $u(t) = \cos(\omega t)$ for $t \geq 0$.

State-space perspective $x(t)$ at $t = 100$. 
Controllability of LTI systems

LTI system of order $n$ in state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$ 

A particular state $x^*$ is **controllable** if we can apply an input $u(t)$ that takes the system from $x(0) = 0$ to $x^*$ in finite time $T$.

**On the board:** Illustrate
Controllability of LTI systems

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**On the board:** Illustrate

- Recall solution of state

\[
x(T) = \int_{t=0}^{T} e^{A(T-\tau)} Bu(\tau) d\tau
\]

  \[
  = \left[ \text{using Cayley-Hamilton’s theorem we get following form} \right] \\
  = B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}
\]
Controllability of LTI systems

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\[
= [\text{using Cayley-Hamilton’s theorem we get following form}]
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\[
= B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}
\]

\Rightarrow a given state $x(T)$ is linear combination of $B, AB, \cdots, A^{n-1}B$. 
Controllability of LTI systems, cont’d

LTI system of order $n$ in state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$  

- $\Rightarrow$ a given state $x(T)$ is linear combination of $B, AB, \ldots, A^{n-1}B$.
- All states $x^*$ are controllable if and only if matrix

$$S(A, B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

has $n$ independent columns. That is, $S$ has full rank.
Controllability of LTI systems, cont’d

LTI system of order $n$ in state-space form

$$\dot{x} = Ax + Bu$$
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$\Rightarrow$ a given state $x(T)$ is linear combination of $B, AB, \ldots, A^{n-1}B$.

All states $x^*$ are controllable if and only if matrix

$$S(A, B) = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

has $n$ independent columns. That is, $S$ has full rank.

System $G$ is controllable $\iff$ all $x^*$ are controllable $\iff$ rank$(S) = n$.

Important property for designing controllers.
Observability of LTI systems

LTI system of order $n$ in state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du.$$ 

- Suppose we have zero input $u(t) \equiv 0$ and initialize system at some state $x(0) = x^* \neq 0$. Then $x^*$ is unobservable if output is unchanged $y(t) \equiv 0$.

**On the board:** Illustrate
Observability of LTI systems

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**On the board:** Illustrate

- If output signal is constant \( y(t) = 0 \), then all derivatives at \( t = 0 \) are

\[
\frac{d^k}{dt^k} y(t)|_{t=0} = C \frac{d^k x(t)}{dt^k}|_{t=0} = CA^k x^* = 0
\]
Observability of LTI systems

LTI system of order $n$ in state-space form

\[
\dot{x} = Ax + Bu \\
y = Cx + Du.
\]

- Constant $y(t) = 0$, means that for observable $x^* \neq 0$ we have

\[
\frac{d^k}{dt^k} y(t) \bigg|_{t=0} = CA^k x^* \neq 0
\]

\[
\Rightarrow Cx^* \neq 0, \quad CAx^* \neq 0, \quad \cdots, \quad CA^{n-1}x^* \neq 0
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\[\Rightarrow Cx^* \neq 0, \quad CAx^* \neq 0, \quad \cdots, \quad CA^{n-1}x^* \neq 0\]

- That is, \( x^* \neq 0 \) is observable if

\[\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x^* \neq 0\]
Observability of LTI systems

LTI system of order $n$ in state-space form
\[ \dot{x} = Ax + Bu \]
\[ y = Cx + Du. \]

- Constant $y(t) = 0$, means that for observable $x^* \neq 0$ we have
  \[ \frac{d^k}{dt^k} y(t) \big|_{t=0} = CA^k x^* \neq 0 \]
  \[ \Rightarrow Cx^* \neq 0, \quad CAx^* \neq 0, \quad \cdots, \quad CA^{n-1}x^* \neq 0 \]
- That is, $x^* \neq 0$ is observable if
  \[ \mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x^* \neq 0 \]

- System $G$ is observable $\Leftrightarrow$ all $x^* \neq 0$ are observable $\Leftrightarrow$
  \[ \text{rank}(\mathcal{O}) = n. \quad (\text{So, } \mathcal{O}x^* = 0 \text{ impossible for } x^* \neq 0) \]
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Observability and controllability

**Why:** Is the system model such that we can observe all state changes through the output signal? Can we affect all the states using our input signal?