
Sensitivity and robustness

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1) When a system is observable
   a the states can be estimated arbitrarily well
   b the states can be controlled arbitrarily well
   c the system is also stable

2) State estimation using an observer
   a does not handle initial errors of the state
   b can be described as a differential equation
   c is an unstable process

3) The transfer function for a control system with estimated states
   a is different from that of control system with known states
   b is the same as that of control system with known states
   c is real-valued
F9: Quiz!

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   b the states can be controlled arbitrarily well ↑
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Sensitivity to disturbance and noise
Control system with disturbances and noise
Using general linear feedback

Closed-loop system using general linear feedback:

\[ G_c(s) = \frac{G(s)F_r(s)}{1 + G(s)F_y(s)} \]

General open-loop system: \( G_o(s) \triangleq F_y(s)G(s) \)
Control system with disturbances and noise
Using general linear feedback

How will the control system cope with unknown disturbances and noise?

[Board: the closed-loop system with $V(s)$ and $N(s)$]
Defining sensitivity functions

▶ Sensitivity function:

\[ S(s) \overset{\triangle}{=} \frac{1}{1 + G_o(s)} \]
Defining sensitivity functions

- **Sensitivity function:**

\[ S(s) \triangleq \frac{1}{1 + G_o(s)} \]

- **Complementary sensitivity function:**

\[ T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)} \]
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- **Complementary sensitivity function:**

\[ T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)} \]

- **Consequence:**

\[ S(s) + T(s) \equiv 1, \quad \forall s \]

- \( S(s) \) and \( T(s) \) affected by controller \( F_y(s) \).
Closed-loop system and sensitivity functions

Closed-loop system:

\[ Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s) \]
Closed-loop system and sensitivity functions

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Want both \(|S(i\omega)|\) and \(|T(i\omega)| \ll 1\) simultaneously...
Closed-loop system and sensitivity functions

- **Closed-loop system:**
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- Want both \(|S(i\omega)|\) and \(|T(i\omega)| \ll 1\) simultaneously...
- ...but **impossible** since

  \[ |S(i\omega)| + |T(i\omega)| \geq |S(i\omega) + T(i\omega)| \equiv 1 \]
Design of poles and zeros via $F_y$ affects also $S$ and $T$
Sensitivity functions in frequency domain
Design trade-off

Example:
- **Disturbance** $v(t)$ with energy at low frequencies
- **Noise** $n(t)$ with energy at high frequencies
Sensitivity functions in frequency domain
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Example:
- **Disturbance** $v(t)$ with energy at low frequencies
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Typical design trade-off is then:
- **low** $\omega$: $|S(i\omega)| \ll 1$ to suppress $V(i\omega)$.
- **high** $\omega$: $|T(i\omega)| \ll 1$ to suppress $N(i\omega)$.
In addition we want Nyquist contour

\[ G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)} \]

far from \(-1\). (Cf. F6 and F7.)
Robustness to model errors
Control systems with model errors

Model $G$ is an approximation

$$r \rightarrow e \rightarrow F \rightarrow u \rightarrow G^0 \rightarrow y^0$$
Control systems with model errors

Model $G$ is an approximation

Assume that the real system can be written as

$$G^0(s) = G(s)(1 + \Delta_G(s))$$

The relative model error of $G(s)$:

$$\Delta_G(s) = \frac{G^0(s) - G(s)}{G(s)}$$
Control systems with model errors

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How is stability of $G^0_c(s)$ affected by unknown error $\Delta_G(s)$?
Model errors and stability
Using the complementary sensitivity function

Assume:

1. Controller $F(s)$ stabilizes the assumed system $G(s)$
2. $G(s)$ and $G^0(s)$ have same number of poles in right half-plane.
3. Open-loop: $F(s)G(s) \to 0$ and $F(s)G^0(s) \to 0$ where $|s| \to \infty$
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(Result 6.2) Robustness criterium
If assumptions are valid and $T(i\omega)$ fulfills

$$|T(i\omega)| < \frac{1}{|\Delta G(i\omega)|}, \quad -\infty \leq \omega \leq \infty$$

$\Rightarrow$ the real closed-loop system $G^0_c(s)$ is also stable!
Model errors and stability
Bounding the model errors

\( \Delta_G(i\omega) \) is unknown but suppose we can cap it by \( g(\omega) > |\Delta_G(i\omega)| \)
Model errors and stability

Bounding the model errors

$\Delta G(i\omega)$ is unknown but suppose we can cap it by $g(\omega) > |\Delta G(i\omega)|$

$|T(i\omega)| < \frac{1}{g(\omega)} < \frac{1}{|\Delta G(i\omega)|}$

$\Rightarrow$ real closed-loop system $G_c^0(s)$ is also stable
Summary and recap

- Sensitivity with respect to disturbances and noise
- Sensitivity functions and their impact on control
- Robustness with respect to model errors
- Robustness criterion in the frequency domain