



# Intro. Computer Control Systems: F10

Sensitivity and robustness

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# F9: Quiz!

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- a the states can be estimated arbitrarily well  $\uparrow$
- b the states can be controlled arbitrarily well  $\uparrow$
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# F9: Quiz!

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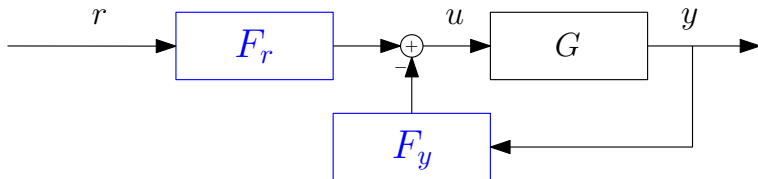
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- 3) The transfer function for a control system with **estimated states**
  - a is different from that of control system with known states  $\uparrow$
  - b is the same as that of control system with known states  $\uparrow$
  - c is real-valued  $\downarrow$



# Sensitivity to disturbance and noise

# Control system with disturbances and noise

## Using general linear feedback



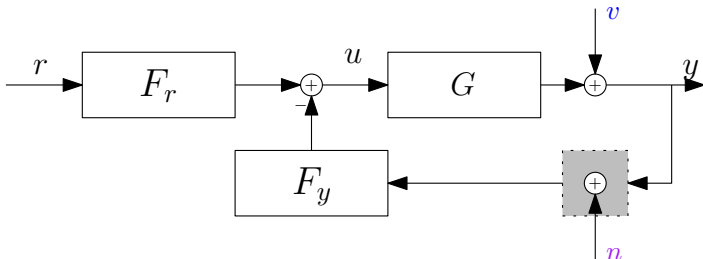
Closed-loop system using general linear feedback:

$$G_c(s) = \frac{G(s)F_r(s)}{1 + G(s)F_y(s)}$$

General open-loop system:  $G_o(s) \triangleq F_y(s)G(s)$

# Control system with disturbances and noise

## Using general linear feedback



How will the control system cope with *unknown* disturbances and noise?

**[Board: the closed-loop system with  $V(s)$  and  $N(s)$ ]**



# Defining sensitivity functions

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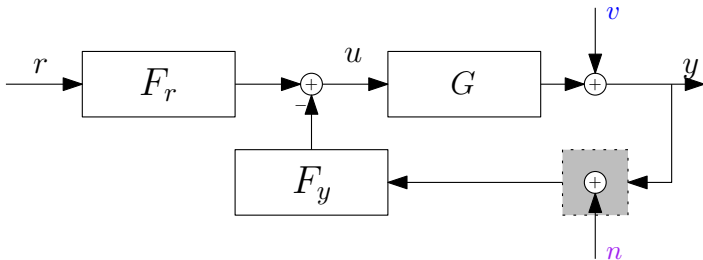
$$T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)}$$

- ▶ **Consequence:**

$$S(s) + T(s) \equiv 1, \quad \forall s$$

- ▶  $S(s)$  and  $T(s)$  affected by controller  $F_y(s)$ .

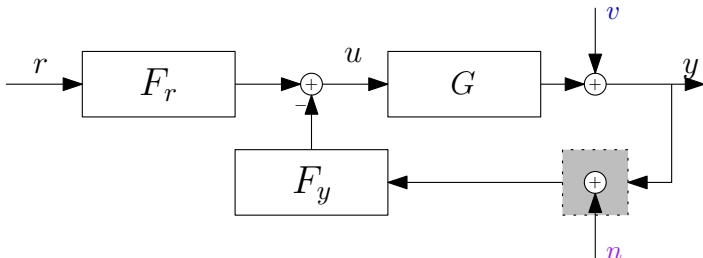
# Closed-loop system and sensitivity functions



- Closed-loop system:

$$Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s)$$

# Closed-loop system and sensitivity functions

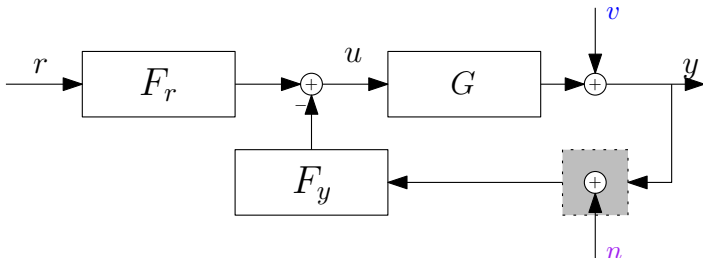


- Closed-loop system:

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- Want both  $|S(i\omega)|$  and  $|T(i\omega)| \ll 1$  *simultaneously*...

# Closed-loop system and sensitivity functions



- ▶ Closed-loop system:

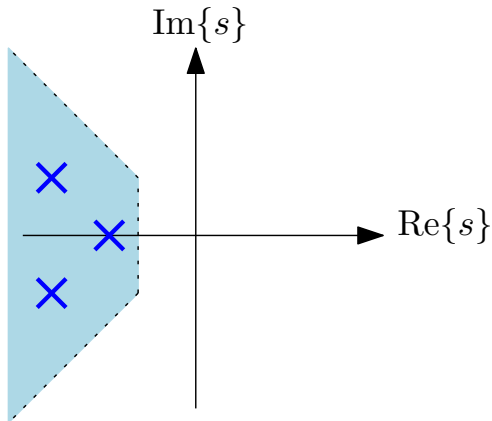
$$Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s)$$

- ▶ Want both  $|S(i\omega)|$  and  $|T(i\omega)| \ll 1$  *simultaneously*...
- ▶ ...but *impossible* since

$$|S(i\omega)| + |T(i\omega)| \geq |S(i\omega) + T(i\omega)| \equiv 1$$

# Design trade-offs

Design of poles and zeros via  $F_y$  affects also  $S$  and  $T$

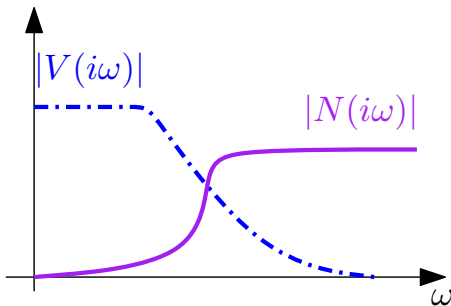


# Sensitivity functions in frequency domain

## Design trade-off

Example:

- ▶ **Disturbance**  $v(t)$  with energy at low frequencies
- ▶ **Noise**  $n(t)$  with energy at high frequencies



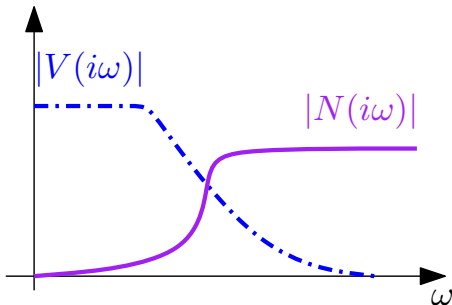


# Sensitivity functions in frequency domain

## Design trade-off

Example:

- ▶ **Disturbance**  $v(t)$  with energy at low frequencies
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Typical design trade-off is then:

- ▶ *low*  $\omega$ :  $|S(i\omega)| \ll 1$  to suppress  $V(i\omega)$ .
- ▶ *high*  $\omega$ :  $|T(i\omega)| \ll 1$  to suppress  $N(i\omega)$ .

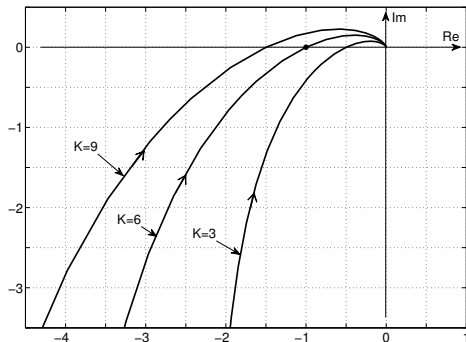
# Sensitivity functions in frequency domain

## Design trade-off

In addition we want **Nyquist contour**

$$G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)}$$

far from **-1**. (Cf. **F6** and **F7**.)

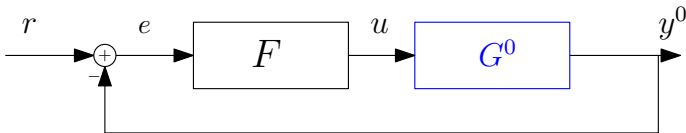




## Robustness to model errors

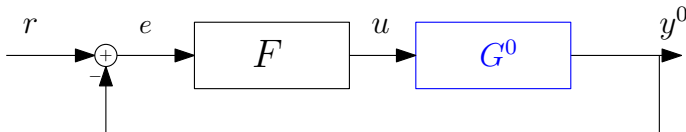
# Control systems with model errors

Model  $G$  is an *approximation*



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- Assume that the **real system** can be written as

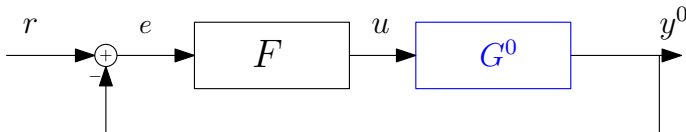
$$G^0(s) = G(s)(1 + \Delta_G(s))$$

- The **relative model error** of  $G(s)$ :

$$\Delta_G(s) = \frac{G^0(s) - G(s)}{G(s)}$$

# Control systems with model errors

Model  $G$  is an *approximation*



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- The **relative model error** of  $G(s)$ :

$$\Delta_G(s) = \frac{G^0(s) - G(s)}{G(s)}$$

- How is stability of  $G_c^0(s)$  affected by unknown error  $\Delta_G(s)$ ?

# Model errors and stability

## Using the complementary sensitivity function

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Assume:

1. Controller  $F(s)$  stabilizes the *assumed* system  $G(s)$
2.  $G(s)$  and  $G^0(s)$  have same number of poles in right half-plane.
3. Open-loop:  $F(s)G(s) \rightarrow 0$  and  $F(s)G^0(s) \rightarrow 0$  where  $|s| \rightarrow \infty$

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### (Result 6.2) Robustness criterium

If assumptions are valid and  $T(i\omega)$  fulfills

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad -\infty \leq \omega \leq \infty$$

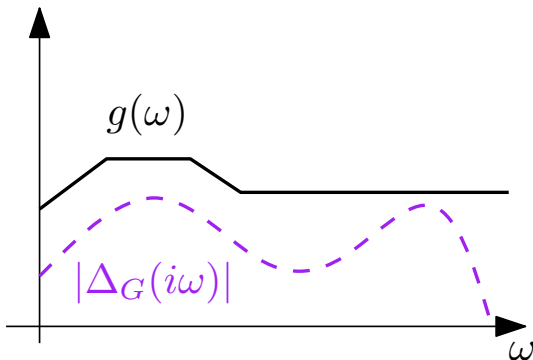
$\Rightarrow$  the *real* closed-loop system  $G_c^0(s)$  is also **stable**!



# Model errors and stability

## Bounding the model errors

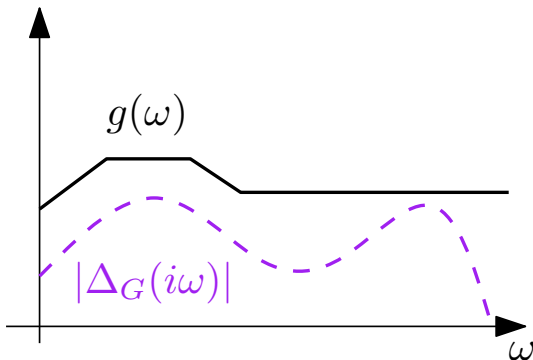
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# Model errors and stability

## Bounding the model errors

$\Delta_G(i\omega)$  is unknown but suppose we can cap it by  $g(\omega) > |\Delta_G(i\omega)|$



$$|T(i\omega)| < \frac{1}{g(\omega)} < \frac{1}{|\Delta_G(i\omega)|}$$

$\Rightarrow$  real closed-loop system  $G_c^0(s)$  is also **stable**

# Summary and recap

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- ▶ Sensitivity with respect to disturbances and noise
- ▶ Sensitivity functions and their impact on control
- ▶ Robustness with respect to model errors
- ▶ Robustness criterion in the frequency domain