



Intro. Computer Control Systems: F2

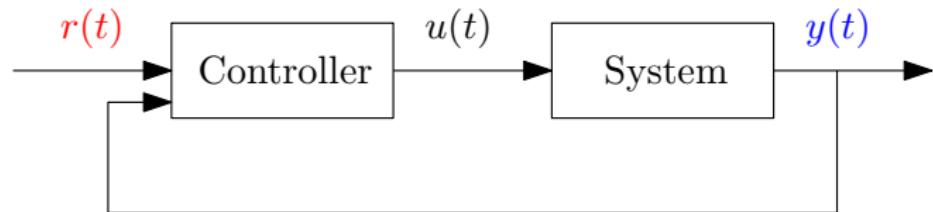
Transfer function, poles and stability

Dave Zachariah

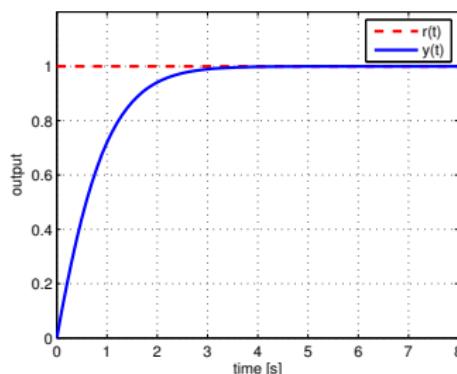
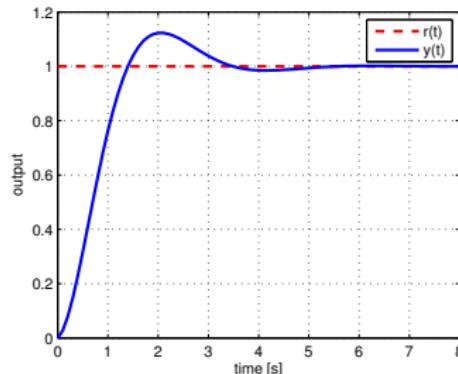
Dept. Information Technology, Div. Systems and Control

F1: Quiz!

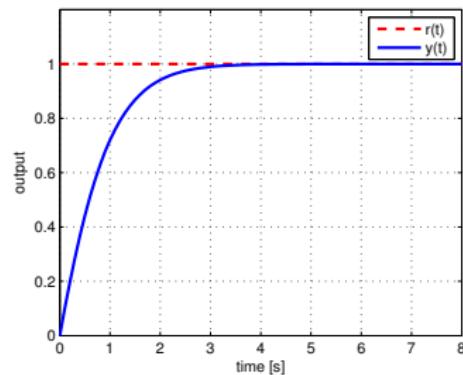
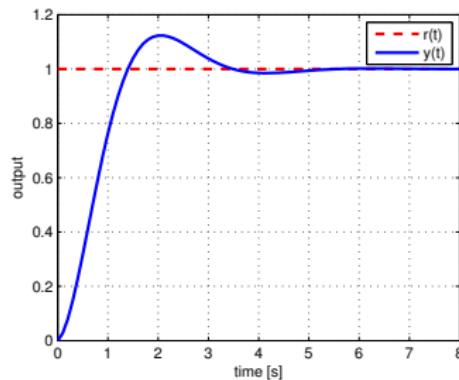
Suppose a control system



comes with two different settings (a) and (b).



F1: Quiz!

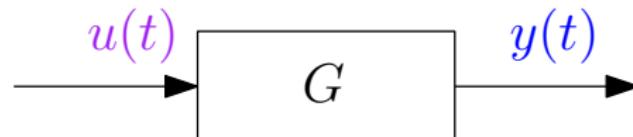


- 1) Which setting of the controller is intuitively better?
- a Setting to the left ↑
 - b Setting to the right ↑
 - c They are equally good ↓



Linear time-invariant system models

Linear time-invariant system models



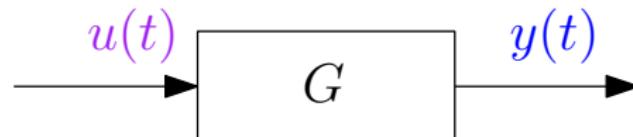
Linear ODE:s are *one possible* input-output description, i.e. of G :

$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_ny = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_mu$$

with initial conditions

Given $u(t)$, solution to ODE = function $y(t)$

Linear time-invariant system models



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Given $u(t)$, solution to ODE = function $y(t)$

Rarely practical in analysis or design for control!

Laplace transform

Used as tool to *solve* and *analyze* linear ODE:s

► **Notation:**

$$y(t) \quad \xleftrightarrow{\mathcal{L}} \quad \mathcal{L}[y(t)] = Y(s)$$

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► **Notation:**

$$y(t) \quad \xleftrightarrow{\mathcal{L}} \quad \mathcal{L}[y(t)] = Y(s)$$

► **Definition:**

$$Y(s) = \mathcal{L}[y(t)] = \int_0^{\infty} y(t)e^{-st}dt, \quad s \in \mathbb{C}$$

Inverse transform:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi i} \int_{\mathbb{C}} Y(s)e^{st}ds, \quad s \in \mathbb{C}$$

Note that s and $Y(s)$ are complex-valued!

Important properties

linearity: $y(t) = \alpha x(t) + \beta z(t) \xleftrightarrow{\mathcal{L}} Y(s) = \alpha X(s) + \beta Z(s)$

derivatives: $\frac{dy}{dt} \xleftrightarrow{\mathcal{L}} sY(s) - y(0)$

$$\frac{d^2y}{dt^2} \xleftrightarrow{\mathcal{L}} s^2Y(s) - sy(0) - \dot{y}(0)$$

⋮

integral: $\int_0^t y(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}Y(s)$

convolution: $\int_0^t x(\tau)z(t-\tau)d\tau \xleftrightarrow{\mathcal{L}} X(s)Z(s)$

final-value thm.*: $\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$

Solving linear ODE with \mathcal{L}

Example: solve **output** $y(t)$

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = 4\frac{d}{dt}u + 5u, \quad u(t), y(0), \dot{y}(0) \quad \text{given}$$

Solving linear ODE with \mathcal{L}

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→ Laplace transform

$$\begin{aligned}\text{LHS} &= (s^2Y(s) - sy(0) - \dot{y}(0)) + 2(sY(s) - y(0)) + 3Y(s) \\ &= (s^2 + 2s + 3)Y(s) - (s + 2)y(0) - \dot{y}(0)\end{aligned}$$

$$\text{RHS} = 4(sU(s) - u(0)) + 5U(s) = (4s + 5)U(s) - 4u(0)$$

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⇒ Set LHS = RHS and solve for $Y(s)$:

$$Y(s) = \frac{4s + 5}{s^2 + 2s + 3}U(s) + \frac{s + 2}{s^2 + 2s + 3}y(0) + \frac{1}{s^2 + 2s + 3}(\dot{y}(0) - 4u(0))$$

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⇒ Given: $U(s) = \mathcal{L}[u(t)]$, $u(0)$, $y(0)$ and $\dot{y}(0)$

Compute $y(t) = \mathcal{L}^{-1}[Y(s)]$ using \mathcal{L}^{-1} -transform (table)



Transfer function and impulse response

Transfer function $G(s)$

- ▶ Assuming initial values are zero $y(0) = \dot{y}(0) = \dots = 0$ and $u(0) = \dot{u}(0) = \dots = 0$. Effect of input u on output y :

$$Y(s) = \underbrace{\frac{4s + 5}{s^2 + 2s + 3}}_{G(s)} U(s),$$

where $G(s)$ is the system transfer function $u \rightarrow y$.

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where $G(s)$ is the system transfer function $u \rightarrow y$.

- ▶ More generally,

$$Y(s) = G(s)U(s)$$

is a model of the *relation* between the system input u and output y .

Transfer function

- ▶ A system described by the linear ODE

$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

with initial values 0.

- ▶ Laplace transform of both sides:

$$(s^n + \cdots + a_{n-1}s + a_n)Y(s) = (b_0s^m + \cdots + b_{m-1}s + b_m)U(s)$$

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$$(s^n + \cdots + a_{n-1}s + a_n)Y(s) = (b_0s^m + \cdots + b_{m-1}s + b_m)U(s)$$

- ▶ System transfer function is a *rational* function:

$$G(s) = \frac{b_0s^m + \cdots + b_m}{s^n + a_1s^{n-1} + \cdots + a_n}$$

Note that s and $G(s)$ are complex-valued!

Impulse response

A system $Y(s) = G(s)U(s)$ (at rest $t = 0$) yields

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \int_0^t g(\tau)u(t - \tau)d\tau,$$

i.e. a *convolution* between $u(t)$ and a system **weighting function**

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

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Suppose input $u(t) = \delta(t)$ = (Dirac)pulse, then output

$$y(t) = \int_0^t g(\tau)\delta(t - \tau)d\tau \equiv g(t).$$

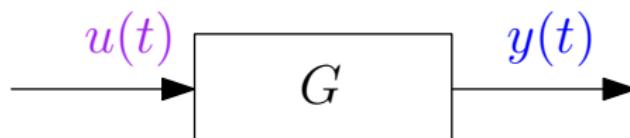
Hence $g(t)$ is called the system **impulse response**.



Poles, zeros and exponential functions

Poles and zeros

Characterizing system behaviour

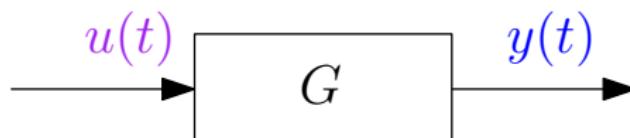


System with transfer function $G(s)$

- ▶ **Zeros:** s' is a *zero*, if $G(s') = 0$.
- ▶ **Poles:** s' is a *pole*, if $G(s')$ is a singularity, that is, $G(s') = \pm\infty$.

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- ▶ **Zeros:** s' is a *zero*, if $G(s') = 0$.
- ▶ **Poles:** s' is a *pole*, if $G(s')$ is a singularity, that is, $G(s') = \pm\infty$.
- ▶ If $G(s) = \frac{B(s)}{A(s)}$ is a *rational function*
 - ▶ zeros = the *roots* to $B(s) = 0$,
 - ▶ poles = the *roots* to $A(s) = 0$.

Poles and solution to linear ODE:s

Characterizing system behaviour

- ▶ Assume model $Y(s) = G(s)U(s)$ with rational transfer function
- ▶ We want

$$y(t) = \int_0^t g(\tau)u(t - \tau)d\tau$$

where $g(\tau) = \mathcal{L}^{-1}[G(s)] = \mathcal{L}^{-1}[B(s)/A(s)].$

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- ▶ Denominator can be factorized by roots (i.e. **poles**):

$$\begin{aligned} A(s) &= s^n + a_1s^{n-1} + \cdots + a_n \\ &= (s + \sigma_1)(s + \sigma_2) \cdots ((s + \sigma_j)^2 + \omega_j^2) \cdots \end{aligned}$$

where **poles** are either

- ▶ real-valued: $-\sigma_1, \dots$
- ▶ complex-conjugated: $-\sigma_j \pm i\omega_j, \dots$

Poles and solution to linear ODE:s

Characterizing system behaviour

- ▶ Express $G(s) = \frac{B(s)}{A(s)}$ using partial-fraction decomposition

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + \sigma_1} + \cdots + \frac{B_j(s)}{(s + \sigma_j)^2 + \omega_j^2} + \cdots$$

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- ▶ Impulse response $g(\tau) = \mathcal{L}^{-1}[G(s)]$ using table:

$$g(t) = b_1 e^{-\sigma_1 t} + \cdots + b_j \sin(\omega_j t + \varphi_j) e^{-\sigma_j t} + \cdots$$

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- ▶ System output:

$$y(t) = \int_0^t g(\tau) u(t - \tau) d\tau = \int_0^t b_1 e^{-\sigma_1 \tau} u(t - \tau) d\tau + \cdots$$

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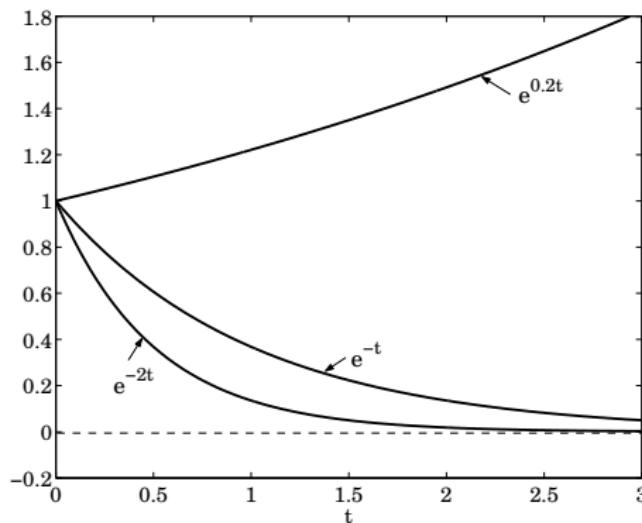
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Output as linear combination of exponential functions

Poles and solution to linear ODE:s

Characterizing system behaviour

Past inputs $u(t)$ integrated by exponential function



Real-parts of poles ($-\sigma$) play an important role



Stability

Stability

Characterizing system behaviour

Definition:

A system $Y(s) = G(s)U(s)$ is **input-output stable** if all bounded inputs $u(t)$ yield a bounded output $y(t)$.

Bounded signal $x(t)$ means $\Leftrightarrow |x(t)| \leq K$ for some K .

[Board: bounded impulse response + real-part of poles]

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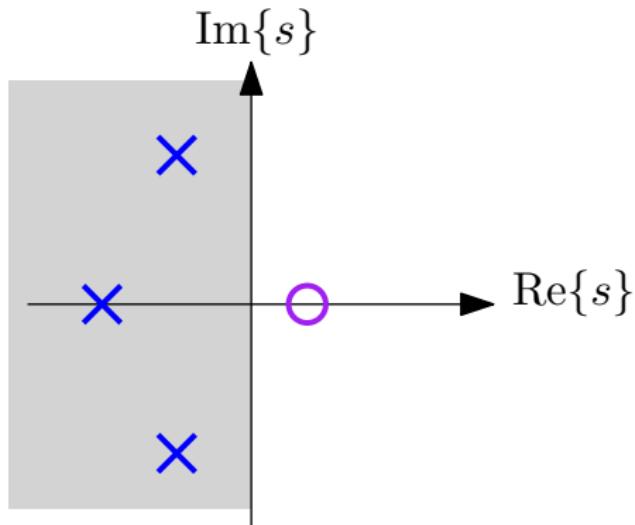
Result:

Assume $G(s) = B(s)/A(s)$ with poles at s_1, s_2, \dots, s_n (and order of denominator \geq numerator)

$$Y(s) = G(s)U(s) \text{ input-output stable} \Leftrightarrow \operatorname{Re}\{s_i\} < 0$$

Graphical representation of poles and zeros

Characterizing system behaviour



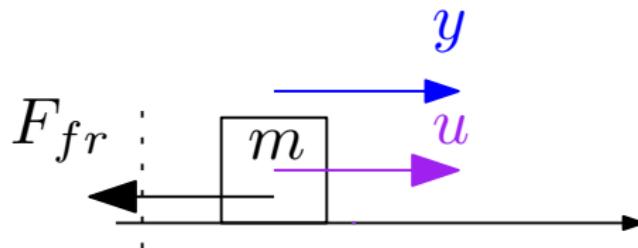
$G(s) = \frac{B(s)}{A(s)}$ stable \Leftrightarrow poles lie in left-halfplane



Examples

Build intuition from simple systems

Ex. #1: Vehicle in motion



Figur: Force $u(t)$ and velocity $y(t)$.

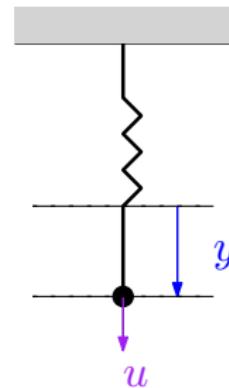
Standard form:

$$\frac{d}{dt}y + \left(\frac{C}{m}\right)y = \left(\frac{1}{m}\right)u$$

[Board: poles]

Build intuition from simple systems

Ex. #2: Damper



Figur: Force $u(t)$ and position $y(t)$.

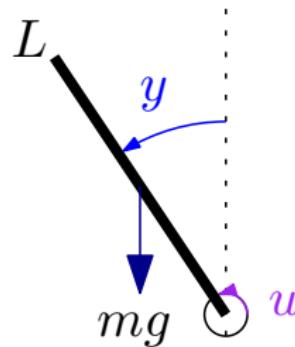
Standard form:

$$\frac{d^2}{dt^2}y + \left(\frac{K}{m}\right)y = \left(\frac{1}{m}\right)u$$

[Board: poles]

Build intuition from simple systems

Ex. #3: Inverted pendulum pendel



Figur: Torque $u(t)$ and angle $y(t)$.

Standard form (around $y \approx 0$):

$$\frac{d^2}{dt^2}y - \left(\frac{3g}{2L}\right)y = \left(\frac{3}{mL^2}\right)u$$

[Board: poles]

Summary and recap

- ▶ Transfer functions as a system description
- ▶ Poles and exponential functions
- ▶ (Bounded) input-output stability