



Intro. Computer Control Systems: F4

PID-controller and its poles, disturbances, root-locus

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F3: Quiz!

- 1) Closed-loop systems $G_c(s)$ with complex-conjugated poles exhibit
- a instability \uparrow
 - b oscillations \uparrow
 - c slow time-response \downarrow

F3: Quiz!

- 1) Closed-loop systems $G_c(s)$ with complex-conjugated poles exhibit
 - a instability ↑
 - b oscillations ↑
 - c slow time-response ↓
- 2) Integral term $\frac{1}{s}$ in a control $F(s)$ can
 - a ensure quick closed-loop system ↑
 - b ensure zero stationary control error ↑
 - c ensure oscillations in the closed-loop system ↓

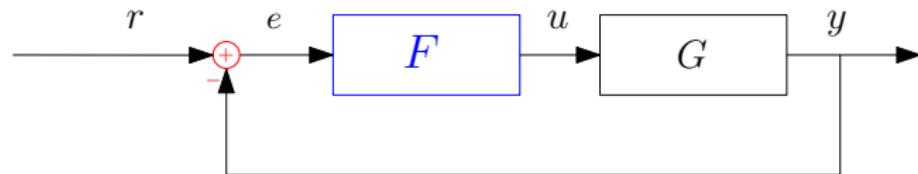


Poles of closed-loop system

Ideal PID-controller

PID-controller with user parameters:

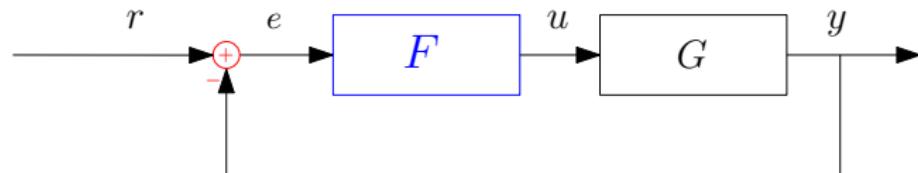
$$u(t) = \underbrace{K_p e(t)}_{P} + \underbrace{\int_{\tau=0}^t e(\tau) d\tau}_{I} + \underbrace{K_d \dot{e}(t)}_{D}$$



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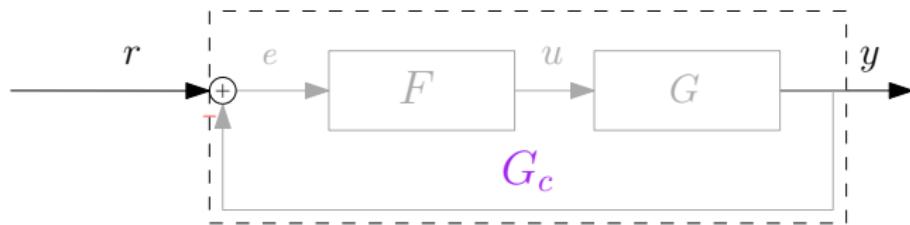


Laplace domain:

$$U(s) = \underbrace{\left(K_p + \frac{K_i}{s} + K_d s \right)}_{\text{controller } F(s)} E(s).$$

Simple feedback control

Closed-loop system $G_c(s)$:



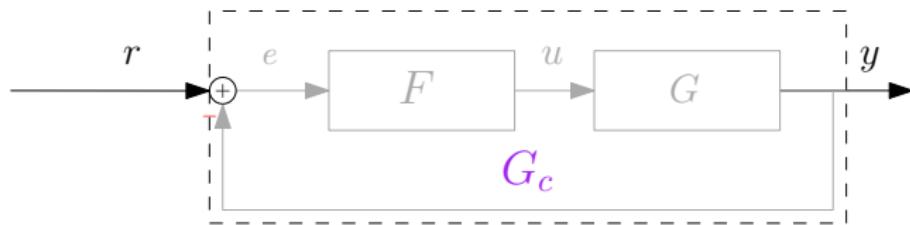
Laplace domain:

$$Y(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}R(s).$$

Determine user parameters K_p , K_i and K_d to obtain
stable closed-loop system

Simple feedback control

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Laplace domain:

$$Y(s) = \frac{G(s)F(s)}{1 + G(s)F(s)} R(s).$$

Determine user parameters K_p , K_i and K_d to obtain
stable closed-loop system
 \Leftrightarrow poles of $G_c(s)$ in left half-plane!



Studying the poles of G_c

Poles of

$$Y(s) = \frac{G(s)F(s)}{1 + G(s)F(s)} R(s).$$

depend on user parameters in $F(s)$.

Classical design methods include:

1. Solve roots of $G_c(s)$ as a function of parameters
2. Routh's algorithm for stability check
3. Root-locus plot for single parameter



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Illustrated in example below



Routh's algorithm

Routh's algorithm

A table to analyze system stability

Assume (closed-loop) system of order n

Poles obtained as roots to polynomial, $a_0 > 0$

$$a_0 s^n + b_0 s^{n-1} + a_1 s^{n-2} + b_1 s^{n-3} + \dots = 0$$

Routh's algorithm

A table to analyze system stability

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Routh's algorithm:

i) Form table

| | | | | |
|-------|-------|-------|-------|---------|
| a_0 | a_1 | a_2 | a_3 | \dots |
| b_0 | b_1 | b_2 | b_3 | \dots |

Routh's algorithm

A table to analyze system stability

Assume (closed-loop) system of order n

ii) Fill the rows ...

| | | | | |
|-------|-------|-------|----------|----------|
| a_0 | a_1 | a_2 | a_3 | \cdots |
| b_0 | b_1 | b_2 | b_3 | \cdots |
| c_0 | c_1 | c_2 | \cdots | |

according to cross rule:

$$c_k = \frac{b_0 a_{k+1} - a_0 b_{k+1}}{b_0}$$

Routh's algorithm

A table to analyze system stability

Assume (closed-loop) system of order n

iii) ...continue

$$\begin{array}{ccccccc} a_0 & a_1 & a_2 & a_3 & \cdots \\ b_0 & b_1 & b_2 & b_3 & \cdots \\ c_0 & c_1 & c_2 & \cdots \\ d_0 & d_1 & d_2 & \cdots \end{array}$$

where again

$$d_k = \frac{c_0 b_{k+1} - b_0 c_{k+1}}{c_0},$$

until one ends with $(n + 1)$ rows

Routh's algorithm

A table to analyze system stability

Assume (closed-loop) system of order n

iv) Check coefficients in the first column

| | | | | |
|----------|-------|-------|---------|---------|
| a_0 | a_1 | a_2 | a_3 | \dots |
| b_0 | b_1 | b_2 | b_3 | \dots |
| c_0 | c_1 | c_2 | \dots | |
| d_0 | d_1 | d_2 | \dots | |
| \vdots | | | | |

Theorem 2.3 (Routh):

If $a_0, b_0, c_0, d_0, \dots > 0 \Leftrightarrow$ all poles lie in the left half-plane (stable system)

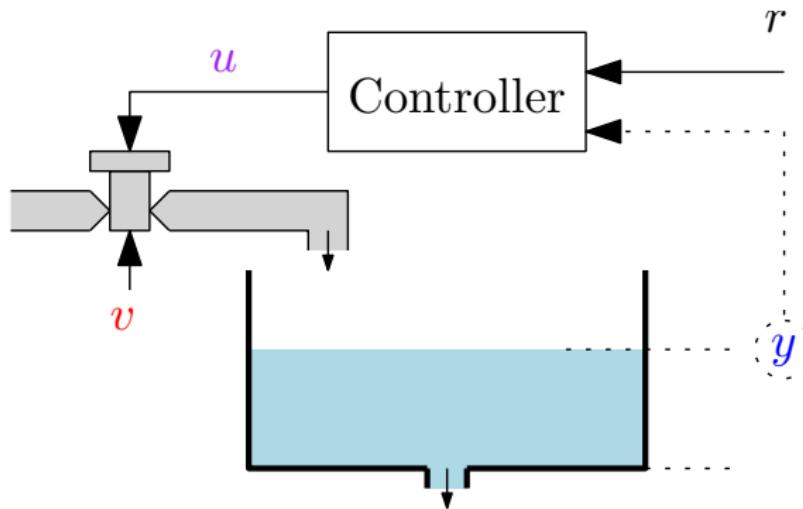
Useful for higher order systems (alt. use Matlab)!



Controlling system with disturbance

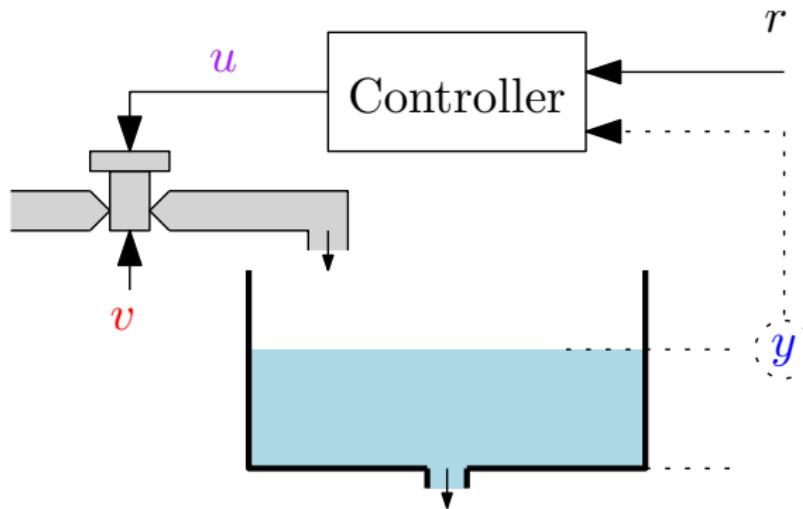
PID-control subject to external disturbance

Example: tank with value + disturbance



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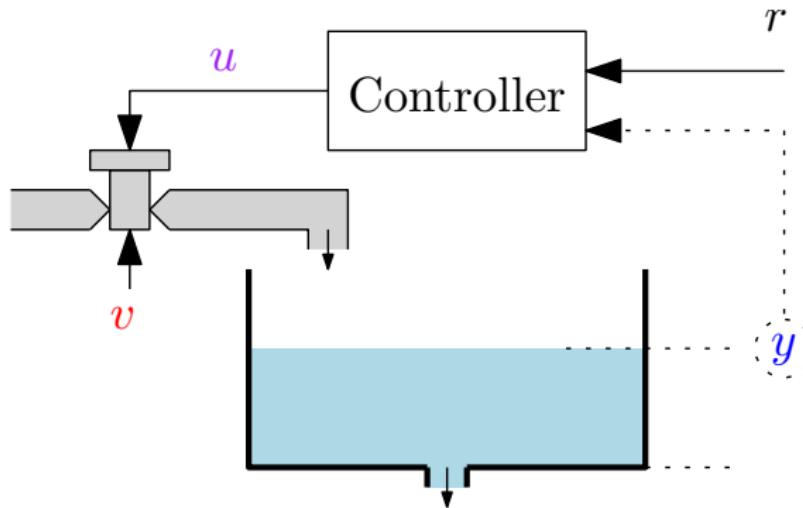


Linear approximate models (around an operating point):

- ▶ Tank (inflow to level): $G_1(s) = \frac{4}{s+1}$
- ▶ Valve (turn angle to ideal outflow): $G_2(s) = \frac{2}{s+2}$

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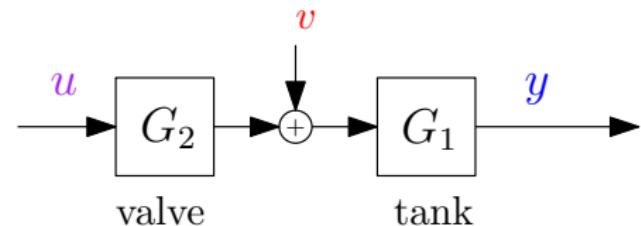
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With pressure disturbance $v(t) \xrightarrow{\mathcal{L}} V(s)$

Example: tank with valve + disturbance

Draw block diagram

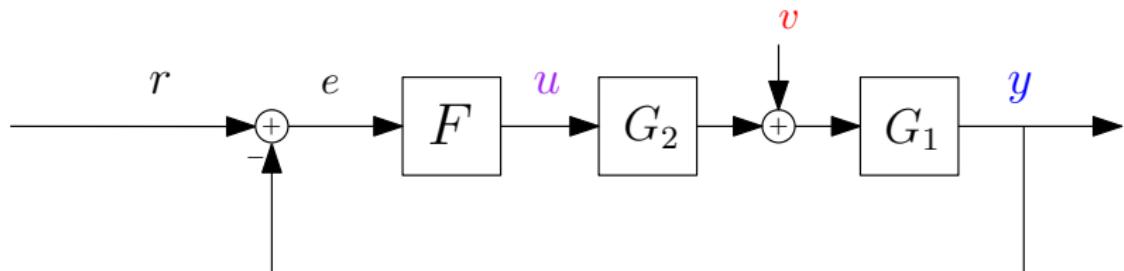
Block diagram:



Example: tank with valve + disturbance

Draw block diagram

Closed-loop system:

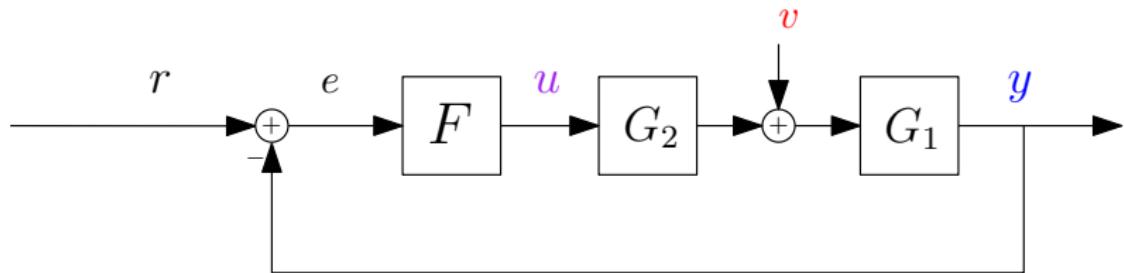


[Board: derive the closed-loop system]

Example: tank with valve + disturbance

Draw block diagram

Closed-loop system:



$$Y(s) = \frac{G_o(s)}{1 + G_o(s)} R(s) + \frac{G_1(s)}{1 + G_o(s)} V(s),$$

where

- ▶ open-loop $G_o(s) \triangleq G_1(s)G_2(s)F(s)$
- ▶ controller $F(s)$



Example: tank with valve + disturbance

Effect of disturbance on control of tank level

Let $r(t) \equiv 0 \quad \xleftarrow{\mathcal{L}} \quad R(s) = 0$ and study effect of **disturbance**

Example: tank with valve + disturbance

Effect of disturbance on control of tank level

Let $r(t) \equiv 0 \quad \xleftarrow{\mathcal{L}} \quad R(s) = 0$ and study effect of **disturbance**

$$Y(s) = 0 + \frac{G_1(s)}{1 + G_1(s)G_2(s)F(s)} V(s),$$

where $F(s) = K_p + \frac{K_i}{s} + K_d s$ is PID-controller.



Example: tank with valve + disturbance

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Insertion yields

$$\begin{aligned} Y(s) &= \frac{\frac{4}{s+1}}{1 + \frac{4}{s+1} \cdot \frac{2}{s+2} \cdot \left(K_p + \frac{K_i}{s} + K_d s \right)} V(s) \\ &= \frac{4s(s+2)}{s^3 + (3 + 8K_d)s^2 + (2 + 8K_p)s + 8K_i} V(s) \end{aligned}$$



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Where do the **poles** end up?

What happens when a sudden constant **disturbance** occurs?



P-control

Example: tank with valve + disturbance

P-control, $K_i = K_d = 0$

Insertion of $K_i = K_d = 0$ gives

$$Y(s) = \frac{4(s+2)}{s^2 + 3s + 2 + 8K_p} V(s)$$

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Closed-loop system poles directly given by:

$$s^2 + 3s + 2 + 8K_p = 0 \quad \Rightarrow s = -1.5 \pm \sqrt{0.25 - 8K_p}$$

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Note choice of parameter K_p :

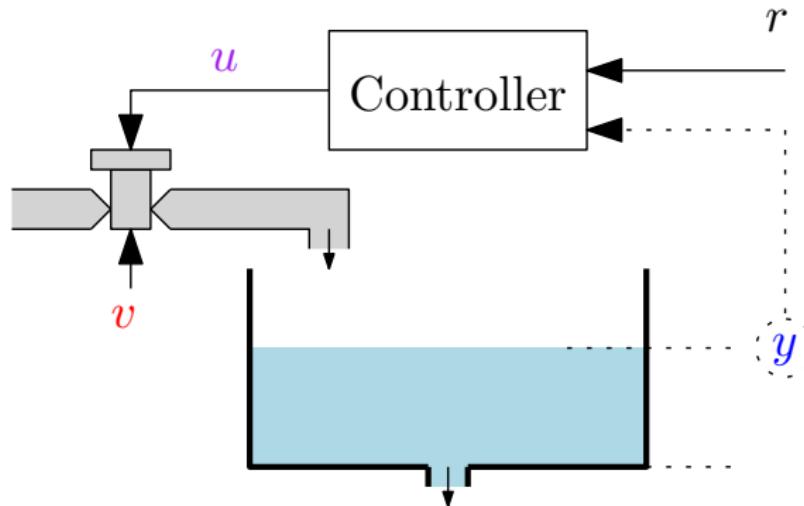
- ▶ $K_p \geq 0$ gives poles in left half-plane
- ▶ $8K_p > 0.25$ or $K_p > 0.03125$ gives complex-conjugated poles

Disturbance suppression

P-control, $K_i = K_d = 0$, when $r(t) \equiv 0$

What happens to tank level $y(t)$ if a *sudden* pressure drop occurs in the pipe? Assume

$$v(t) = \begin{cases} -v_0 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \xleftrightarrow{\mathcal{L}} \quad V(s) = -\frac{v_0}{s}$$



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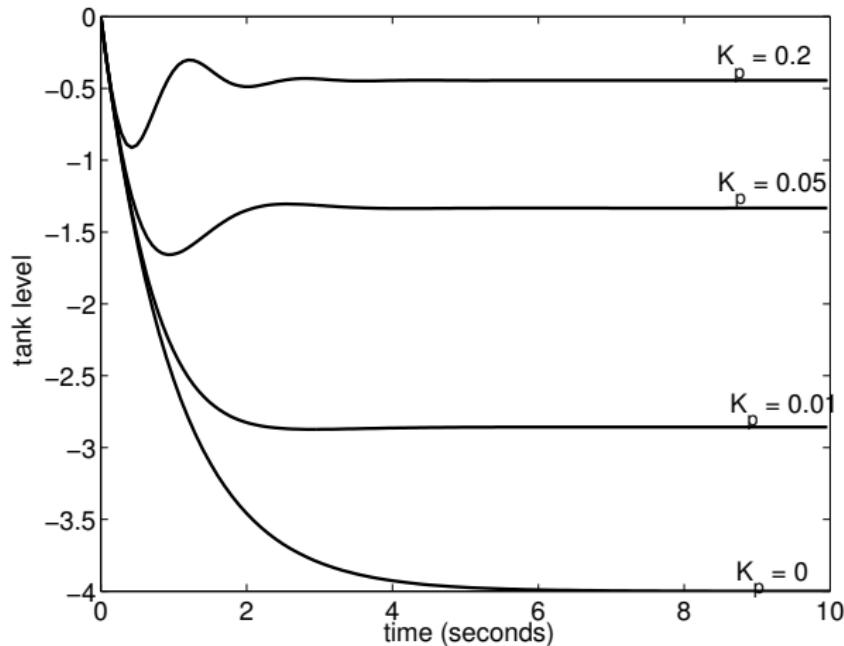
When closed-loop system from $v(t)$ to $y(t)$ is stable: Use *final-value theorem*

$$y_f = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$$

[Board: derive final value of the closed-loop system]

Disturbance suppression

P-control, $K_i = K_d = 0$, when $r(t) \equiv 0$



Note stability, quickness, final value and when oscillations occur



PI-control

Example: tank with valve + disturbance

PI-control, $K_d = 0$

Insertion of $K_d = 0$ gives

$$Y(s) = \frac{4s(s+2)}{s^3 + 3s^2 + (2 + 8K_p)s + 8K_i} V(s)$$

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Using Routh's algorithm: i) Tabulate

| | | |
|---|------------|---|
| 1 | $2 + 8K_p$ | 0 |
| 3 | $8K_i$ | 0 |

Example: tank with valve + disturbance

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Routh's algorithm: ii) Fill rows...

| | | |
|------------------------|------------|---|
| 1 | $2 + 8K_p$ | 0 |
| 3 | $8K_i$ | 0 |
| $(24K_p + 6 - 8K_i)/3$ | 0 | 0 |

Example: tank with valve + disturbance

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Routh's algorithm: iv) Check coefficients in the first column

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| 8K _i | | 0 |

To obtain poles in left half-plane:

$$\Leftrightarrow 24K_p + 6 - 8K_i > 0 \quad \text{och} \quad K_i > 0$$

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Choice of parameters K_p and K_i :

- $0 < K_i < \frac{3}{4} + 3K_p$ gives a stable closed-loop system.

Disturbance suppression, cont'd.

PI-control, $K_d = 0$, when $r(t) \equiv 0$

What happens to tank level $y(t)$ if a *sudden* pressure drop occurs in the pipe? Assume

$$v(t) = \begin{cases} -v_0 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \xleftrightarrow{\mathcal{L}} \quad V(s) = -\frac{v_0}{s}$$

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$$\begin{aligned} y_f &= \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) \\ &= \lim_{s \rightarrow 0} s \frac{4s(s+2)}{s^3 + 3s^2 + (2 + 8K_p)s + 8K_i} \frac{-v_0}{s} \\ &= \frac{-4 \cdot 0 \cdot v_0}{8K_i} = 0! \end{aligned}$$

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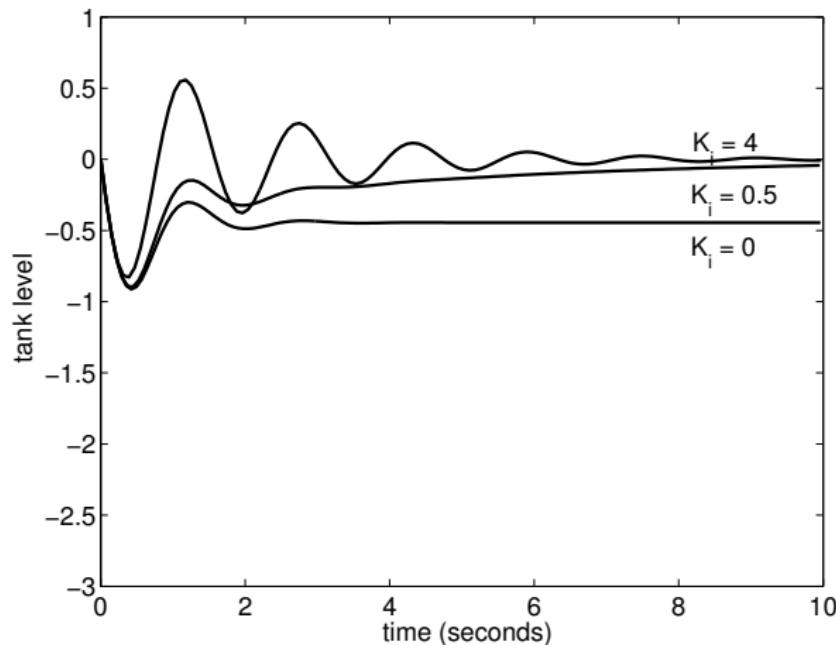
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Effect of constant **disturbance** vanishes with integral!

Disturbance suppression, cont'd.

PI-control, $K_d = 0$, when $r(t) \equiv 0$



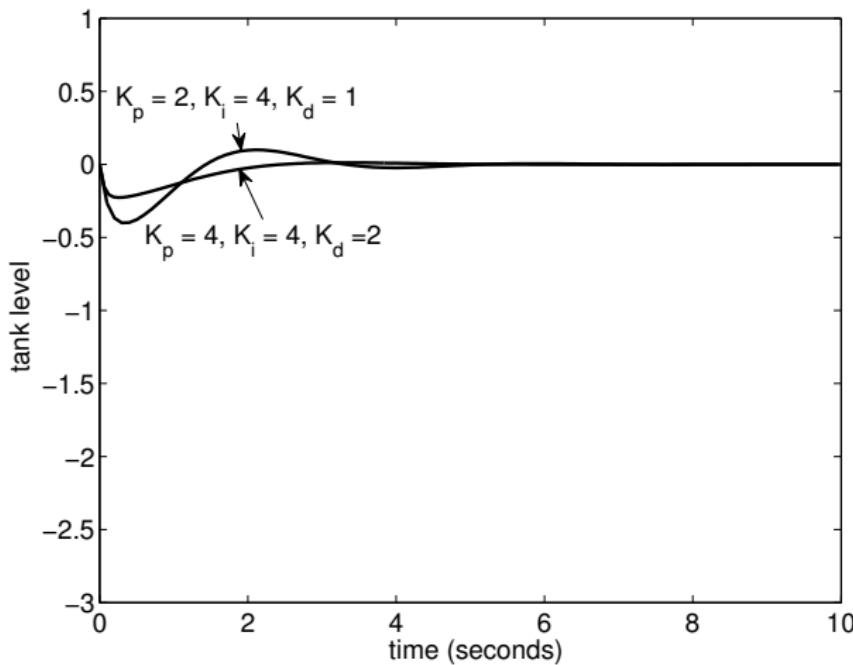
Note stability and final value



PID-control

Disturbance suppression, cont'd.

PID-control, when $r(t) \equiv 0$



PID-control gives in *this case* free choice of poles using K_p, K_i and K_d



Root locus

Example: PI-control of a simple system

Poles on \mathbb{C} -plane as a function of single user-parameter

Assume system $Y(s) = \underbrace{\frac{1}{s+1}}_{G(s)} U(s)$ with PI-controller:

$$U(s) = \underbrace{\left(K + \frac{10K}{s} \right)}_{F(s)} (R(s) - Y(s)) \quad (K_p = K, K_i = 10K)$$

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gives closed-loop system

$$\begin{aligned} Y(s) &= \frac{G(s)F(s)}{1 + G(s)F(s)} R(s) = \frac{K \frac{s+10}{s(s+1)}}{1 + K \frac{s+10}{s(s+1)}} R(s) \\ &= \frac{K(s+10)}{s(s+1) + K(s+10)} R(s) \end{aligned}$$

with single parameter K

Root locus

A graphical method to study how poles depend on K

Closed-loop system can be written on the form:

$$G_c(s) = \frac{\dots}{P(s) + KQ(s)}$$

A root locus shows the roots to the polynomial equation

$$P(s) + KQ(s) = 0, \quad 0 \leq K < \infty \quad (*)$$

as a function of parameter K .

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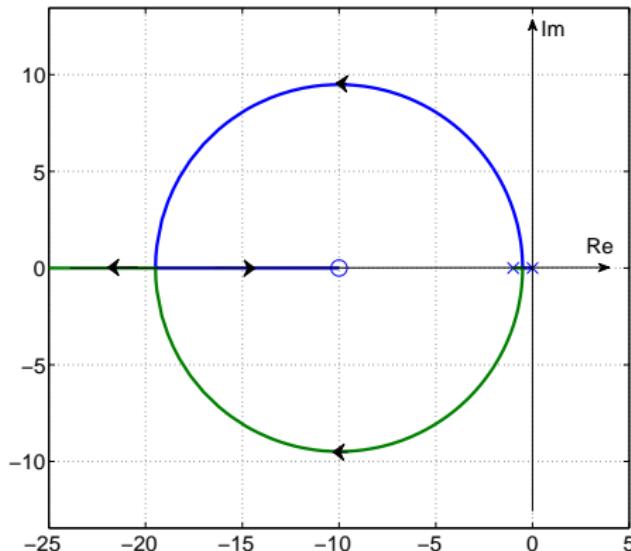
Definition:

Root locus = the points s that solve $P(s) + KQ(s) = 0$ (roots/poles) as we vary K .

Root locus

Example: PI-control of simple system

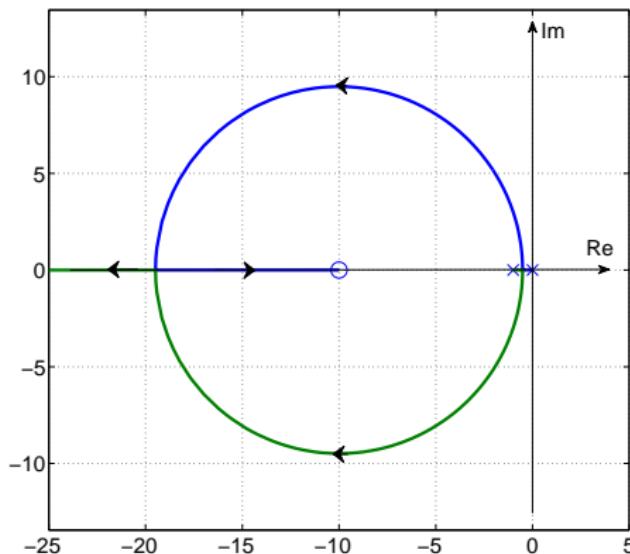
Poles of $G_c(s)$ change when K goes from 0 to ∞ :



Root locus

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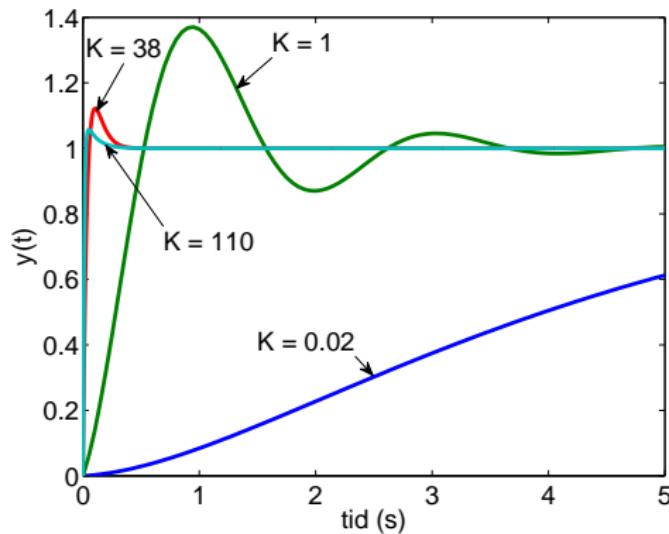


How do we expect closed-loop system to behave at different K ?

Corresponding step responses

Exemple: PI-control of simple system

Closed-loop system response to step $R(s) = \frac{1}{s}$ for different K :



Characteristics of root locus

- ▶ Polynomial $P(s)$ and $Q(s)$ have order n and m , resp.
- ▶ Equation $(*)$ has always n roots, which constitute n *branches* in the root locus.
- ▶ $P(s)$ and $Q(s)$ have real-valued coefficients \Rightarrow complex-valued roots to $(*)$ are all complex-conjugated pairs \Rightarrow root locus is *symmetric around the real-axis*.

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- ▶ The root locus is characterized by:
 1. starting points
 2. end points
 3. asymptotes
 4. parts on the real-axis

Characteristics of root locus, cont'd.

Results 3.1 and 3.2

Characteristics of root locus, cont'd.

Results 3.1 and 3.2

- ▶ **Starting point:** The n roots to $(*)$ for $K = 0$, given by $P(s) = 0$. Use 'x'.

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- ▶ **End points:** The m finite roots to $(*)$ when $K \rightarrow \infty$, given by $Q(s) = 0$. Use 'o'.

Characteristics of root locus, cont'd.

Results 3.1 and 3.2

- ▶ **Starting point:** The n roots to $(*)$ for $K = 0$, given by $P(s) = 0$. Use 'x'.
- ▶ **End points:** The m finite roots to $(*)$ when $K \rightarrow \infty$, given by $Q(s) = 0$. Use 'o'.
- ▶ **Asymptotes:** $n - m$ branches that extend to infinity. The asymptotes extend (symmetrically) from a point on the real axis.

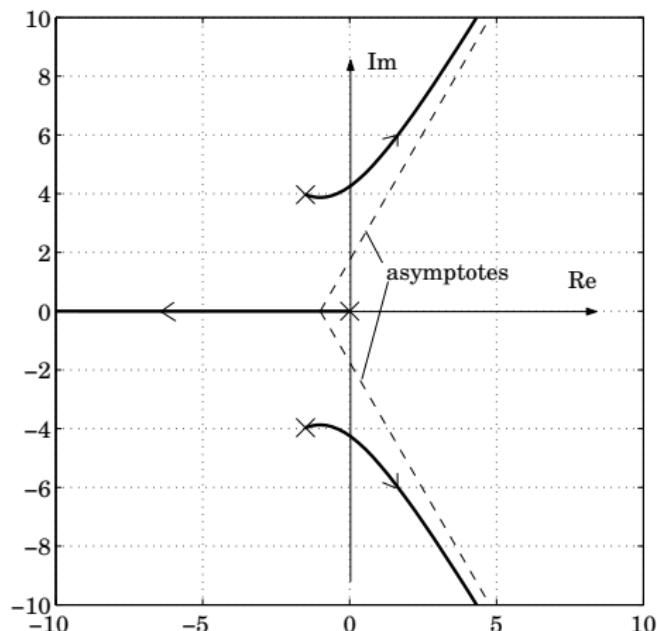
Characteristics of root locus, cont'd.

Results 3.1 and 3.2

- ▶ **Starting point:** The n roots to $(*)$ for $K = 0$, given by $P(s) = 0$. Use 'x'.
- ▶ **End points:** The m finite roots to $(*)$ when $K \rightarrow \infty$, given by $Q(s) = 0$. Use 'o'.
- ▶ **Asymptotes:** $n - m$ branches that extend to infinity. The asymptotes extend (symmetrically) from a point on the real axis.
- ▶ **Real axis:** The parts of the real-axis with an odd number of real-valued starting- and end points to their right, belong to the root locus.
- ▶ **Branches** of root locus cannot cross. Thus if two branches meet on the real axis they must 'split' in the complex plane.

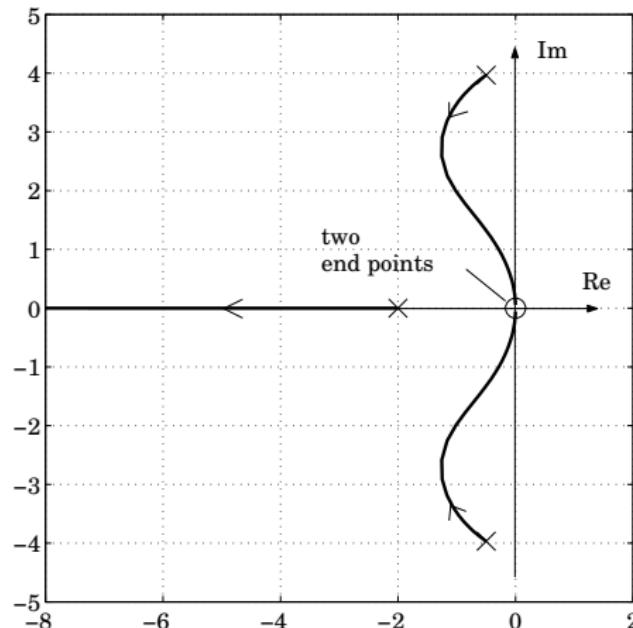
PI-control of the tank

Root locus for $G_c(s)$ with respect to K_i (fix $K_p = 2$ and $K_d = 0$).



PID-control of the tank

Root locus for $G_c(s)$ with respect to K_d (fix $K_p = 2$ and $K_i = 4$).



Summary and recap

- ▶ PID-controller and closed-loop system stability
- ▶ Example of PID-controller and disturbance suppression
- ▶ Closed-loop system poles as a function of parameters K