Intro. Computer Control Systems:
F6
Bode plot, design in frequency domain, Nyquist curve, minimum phase

Dave Zachariah

Dept. Information Technology, Div. Systems and Control
1) **General** feedback control
   a. Leads to unstable closed-loop systems ↑
   b. Leads to more design freedom ↑
   c. Leads to non-minimum phase systems ↓
F5: Quiz!

1) General feedback control
   a. Leads to unstable closed-loop systems ↑
   b. Leads to more design freedom ↑
   c. Leads to non-minimum phase systems ↓

2) For linear time-invariant systems a sinusoidal input yields
   a. A sinusoidal output ↑
   b. A exponentially declining output ↑
   c. A stable output ↓
Frequency response of system
Properties in the frequency domain

How does a (closed-loop) system respond to oscillating signals?

\[ r(t) = \frac{1}{2\pi} \int R(i\omega) e^{i\omega t} d\omega, \]

where \( e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \)

\( \alpha e^{i(\omega t + \phi)} e^{i\omega t} \)

\[ \Rightarrow \]

System output \( y(t) \) is reweighted sum of the input cosine- and sine signals!
Properties in the frequency domain

How does a (closed-loop) system respond to oscillating signals?

- Input signals decomposed into sum of cosine- and sine signals:

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where

\[ e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \quad \text{and} \quad \omega = 2\pi f \quad \text{(frequency)} \]
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- \( e^{i\omega t} \) is an eigen-function to linear time-invariant systems:

\[ e^{i\omega t} \xrightarrow{G} \alpha e^{i(\omega t + \phi)} \]

\[ \text{System output } y(t) \text{ is reweighted sum of the input cosine- and sine signals!} \]
Properties in the frequency domain

How does a (closed-loop) system respond to oscillating signals?

- Input signals decomposed into sum of cosine- and sine signals:

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- \( e^{i\omega t} \) is an eigen-function to linear time-invariant systems:

\[ e^{i\omega t} \rightarrow G \rightarrow \alpha e^{i(\omega t+\phi)} \]

- \( \Rightarrow \) System output \( y(t) \) is reweighted sum of the input cosine- and sine signals!
Frequency response
Example of (closed-loop) system

\[ Y(s) = G_c(s)R(s) \]
Frequency response
Example of (closed-loop) system

Frequency response: \[ G_c(s) \bigg|_{s=i\omega} = G_c(i\omega) \]

Example: Magnitude curve of frequency response
Frequency response
Example of (closed-loop) system

Signal as a weighted sum of cosine- and sine signals:

\[ r(t) = \frac{1}{2\pi} \int R(i\omega)e^{i\omega t} \, d\omega. \]

Example: Frequency content of signal

\[ |R(i\omega)| \]

Note that the plot is symmetric \(|X(i\omega)| = |X(-i\omega)|\)
Frequency response

Example of (closed-loop) system

Signal as a weighted sum of cosine- and sine signals:

\[ y(t) = \frac{1}{2\pi} \int Y(i\omega) e^{i\omega t} d\omega. \]

Example: Frequency content of output

\[ |Y(i\omega)| = |G_c(i\omega)| |R(i\omega)| \]
Frequency response
Example of (closed-loop) system

\[ |R(i\omega)| \]

\[ |Y(i\omega)| = |G_c(i\omega)||R(i\omega)| \]

Cf. “sine-in sine-out”
Frequency response
Example of (closed-loop) system

\[ |Y(i\omega)| = |G_c(i\omega)| |R(i\omega)| \]

Goal: \[ y(t) \approx r(t) \iff Y(i\omega) \approx R(i\omega) \]

- Closed-loop system and frequency response:
  \[ Y(i\omega) = G_c(i\omega) R(i\omega) = |G_c(i\omega)| e^{i\arg\{G_c(i\omega)\}} R(i\omega) \]

- Ideal: Magnitude \( |G_c(i\omega)| \approx 1 \) and phase \( \arg\{G_c(i\omega)\} \approx 0 \)
Sketching the frequency response
Poles/zeros and frequency response
Bode plot: recipe for magnitude curve

Re-write *standard form* using the poles and zeros:

\[
G(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = K_0 \frac{(s + z_1) \cdots (s + z_m)}{(s + p_1) \cdots (s + p_n)}
\]
Re-write on modified form:

\[
G(s) = \cdots = K_0 \frac{(s + z_1) \cdots (s + z_m)}{(s + p_1) \cdots (s + p_n)} = K \frac{(1 + \frac{s}{z_1}) \cdots (1 + \frac{s}{z_m})}{s^q(1 + \frac{s}{p_1}) \cdots (1 + \frac{s}{p_n})}
\]
Poles/zeros and frequency response

Bode plot: recipe for magnitude curve

System on modified form:

\[ G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots}{s^q(1 + \frac{s}{p_1}) \cdots} \]

**Logarithm of \(|frequency response|\):**

\[
\log_{10} |G(i\omega)| = \log_{10} |K| - q \log_{10} |\omega| + \log_{10} |1 + \frac{i\omega}{z_1}| + \cdots - \log_{10} |1 + \frac{i\omega}{p_1}| - \cdots
\]

**Intuition:** Each term is turned on as \( \omega \) increases

- when \( \omega \ll |z_k|, |p_k| \), the term is \( \approx 0 \)
- when \( \omega \gg |z_k|, |p_k| \), the term increases/decreases with \( \omega \)
Poles/zeros and frequency response
Bode plot: recipe for magnitude curve

Recipe for sketching magnitude curve

\[ \log_{10} |G(i\omega)| \]

1. Compute all \( z_k \) and \( p_k \)
2. Sort \( |z_k| \) or \( |p_k| \) by distance to the origin.
3. Evaluate \( \log |G(i\omega)| \) at the first \( |z_1| \) or \( |p_1| \) after the origin.
4. Plot the curve along \( \omega \to \infty \):
   - Each zero \( \omega \gg |z_k| \) increases the slope by +1.
   - Each pole \( \omega \gg |p_k| \) decreases the slope by −1.
   - Complex-conjugated poles give resonance peak at \( \omega \approx |p_k| \) if \( \zeta \ll 1 \).
Bode plot

Example

1:st order system with simple poles.

Examples:

\[ G_1(s) = \frac{0.2}{s + 0.2}, \]
\[ G_2(s) = \frac{1}{s + 1}, \]
\[ G_3(s) = \frac{5}{s + 5}, \]
\[ (F_{PD}(s) = 1 + s) \]
Bode plot

Example

2nd order system with complex conjugated poles:

\[ G(s) = \frac{\omega_0^2}{s^2 + 2\zeta \omega_0 s + \omega_0^2} \]

Poles \(-\omega_0 \zeta \pm i\omega_0 \sqrt{1 - \zeta^2}\) där \(|p_1| = |p_2| = \omega_0\).

\[ \zeta \ll 1 \Rightarrow \text{gives resonance peak} \geq |G(i\omega_0)| = \frac{1}{2\zeta} \]
Bode plot

Example

\[ G(s) = \frac{100(s + 1)}{s(s^2 + 6s + 100)} \]

- **Zeros:** \(-1\)
- **Poles:**

\[ 0 \quad \text{and} \quad -3 \pm i\sqrt{91} \]

where \(\omega_0 = 10\) and \(\zeta = 0.3\).

[Board: sketch magnitude curve]
Bode plot
Example

\[ G(s) = \frac{100(s + 1)}{s(s^2 + 6s + 100)} \]

- **Zeros:** \(-1\)
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  \[ 0 \quad \text{and} \quad -3 \pm i\sqrt{91} \]

where \( \omega_0 = 10 \) and \( \zeta = 0.3 \).

[Board: sketch magnitude curve]
Poles/zeros and frequency response

Bode plot: recipe for phase curve

System on modified form:

\[ G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots}{s^q(1 + \frac{s}{p_1}) \cdots} \]

Argument of frequency response (in radians):

\[ \text{arg}\{G(i\omega)\} = -q \cdot \pi + \text{arctan} \frac{\omega}{z_1} + \cdots - \text{arctan} \frac{\omega}{p_1} - \cdots \]
Poles/zeros and frequency response

Bode plot: recipe for phase curve

System on *modified* form:

\[ G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots}{s^q(1 + \frac{s}{p_1}) \cdots} \]

**Argument** of frequency response (in radians):

\[ \arg\{G(i\omega)\} = -q \cdot \pi + \arctan\frac{\omega}{z_1} + \cdots - \arctan\frac{\omega}{p_1} - \cdots \]

**Plot phase curve** If \( G(s) \) does not have any poles/zeros in right half-plane:

- \( \omega \rightarrow 0 \): \( \arg G(i\omega) \rightarrow 0 \) \( \text{i ng} \frac{1}{s} \) and \( G(0) > 0 \)
- \( \omega \rightarrow \infty \): each zero or pole contributes with \( +\frac{\pi}{2} \) or \( -\frac{\pi}{2} \) in phase, respectively.
- Each pole at the origin contributes with \( -\frac{\pi}{2} \) in phase \( \forall \omega \)

Zeros in right half-plane gives negative phase contribution.
Controller design in time vs. frequency domain
Design principles for controller

Feedback control system:

- Different performance metrics for $G_c$
- Different design principles of $G_c$ via $F$
Design principles for controller

Feedback control system:

- Different performance metrics for $G_c$
- Different design principles of $G_c$ via $F$
- Ideal controller in the frequency domain:

$$|G_c(i\omega)| \approx 1 \quad \text{and} \quad \arg\{G_c(i\omega)\} \approx 0$$

for the relevant frequencies $\omega$. 
Controller design in time domain

Specifications of the time response

\[ y(t) \text{ when } r(t) \text{ is a step } \iff R(s) = \frac{r_0}{s} : \]
Controller design in time domain

Specifications of the time response

\( y(t) \) when \( r(t) \) is a step \( \iff R(s) = \frac{r_0}{s} \):

Performance metrics

- Quickness: rise time \( T_r = t_{90\%} - t_{10\%} \)
- Damping: overshoot \( M = (y_{\text{max}} - y_f)/y_f \)
- Accuracy: static control error \( e_f = r_0 - y_f \) (see F3!)
Controller design in frequency domain

Specifications of the frequency response

\[ |Y(i\omega)| = |G_c(i\omega)| |R(i\omega)| \]

- **Performance metrics**
  - **Quickness:** bandwidth \( \omega_B \) where \( |G_c(i\omega_B)| = \frac{|G_c(0)|}{\sqrt{2}} \)
  - **Damping:** resonance peak level \( M_p = \max(|G_c(i\omega)|) \)
  - **Accuracy:** static gain \( G_c(0) \) (see F3!)

\[ \omega_B \]
Controller design in frequency domain

Specifications of the frequency response

\[ |Y(i\omega)| = |G_c(i\omega)| |R(i\omega)| \]

\[ |G_c(i\omega)| \]

\[ |G_c(0)| \]

\[ \sqrt{2} |G_c(0)| \]

\[ \omega_B \]

Performance metrics

- **Quickness**: bandwidth \( \omega_B \) where \( |G_c(i\omega_B)| = |G_c(0)| / \sqrt{2} \)
- **Damping**: resonance peak level \( M_p = \max(|G_c(i\omega)|) \)
- **Accuracy**: static gain \( G_c(0) \) (see F3!)
Frequency-based design via open-loop system
Design $G_c$ via open-loop system $G_o$

Frequency response and Nyquist curve

Simple feedback control system:

$\text{Closed-loop: } G_c(s) = \frac{G_o(s)}{1+G_o(s)}$

$\text{Open-loop: } G_o(s) = F(s)G(s)$
Design $G_c$ via open-loop system $G_o$

Frequency response and Nyquist curve

Simple feedback control system:

- **Closed-loop:** $G_c(s) = \frac{G_o(s)}{1 + G_o(s)}$
- **Open-loop:** $G_o(s) = F(s)G(s)$
- $G_c(s)$ stable $\iff$ $1 + G_o(s)$ no roots in right half-plane.
Nyquist curve
Example

<table>
<thead>
<tr>
<th>Nyquist curve</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Plot</strong> complex $G_o(i\omega) = G(i\omega)F(i\omega)$, over $0 \leq \omega &lt; \infty$.</td>
</tr>
</tbody>
</table>
Nyquist curve

**Example**

Plot complex $G_o(i\omega) = G(i\omega)F(i\omega)$, over $0 \leq \omega < \infty$.

DC-motor $G(s) = \frac{1}{s(s+1)}$ with controller $F(s) = \frac{K}{s+2}$

Then $G_c$: $K = 3$ (stable), $K = 9$ (unstable), and $K = 6$ (marginally stable).
(Result 3.3) Basic Nyquist criterion:

If $G_o(s)$ has no poles in right half-plane:

$G_c(s)$ stable $\iff$ Nyquist curve $G_o(i\omega)$ does not encircle $-1$
Nyquist criterion

In general: Let $s = \sigma + i\omega$ form semi-circle $\gamma$, with growing radius $R \to \infty$. Then Nyquist curve $G_o(s)$ is $\gamma'$. 
Nyquist criterion

*In general:* Let \( s = \sigma + i\omega \) form semi-circle \( \gamma \), with growing radius \( R \to \infty \). Then Nyquist curve \( G_o(s) \) is \( \gamma' \).

\[
\text{#poles}\{G_c(s)\} \text{ in right half-plane} = \quad \text{#poles}\{G_o(s)\} \text{ in right half-plane} + \text{Number of positive circles of } \gamma' \text{ around } -1
\]
Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

$$|G_c(i\omega)| = \frac{|G_o(i\omega)|}{|1 + G_o(i\omega)|} = \frac{|G_o(i\omega)|}{|G_o(i\omega) - (-1)|}$$

distance to $-1$

Nyquist curve $G_o(i\omega)$

$dave.zachariah@it.uu.se$
Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

Nyquist curve $G_o(i\omega)$:

Crossover frequency and phase margin

Find $\omega = \omega_c$ where $|G_o(i\omega_c)| = 1$ and then define

$$\varphi_m = \arg\{G_o(i\omega_c)\} + 180^\circ$$
Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

Open-loop design $G_o(i\omega)$:

- Crossover frequency $\omega_c \Rightarrow$ bandwidth $\omega_B$ of $G_c$ (quickness)
- Phase margin $\varphi_m \Rightarrow$ resonance peak $M_p$ of $G_c$ (damping)
Controller design via Nyquist/Bode plot

Specification via $G_o(s)$

Open-loop frequency characteristics that we can shape:

- Crossover frequency $\omega_c$
- Phase margin $\varphi_m$

![Graph showing Bode plots]

**Design:**

$$\log_{10} |G_o| = \log_{10} |G| + \log_{10} |F|$$

$$\arg\{G_o\} = \arg\{G\} + \arg\{F\}$$
Compensate P-controller in frequency domain

Control structure starting with P-control:

\[ F(s) = K \underbrace{F_{\text{lead}}(s)F_{\text{lag}}(s)}_{\text{P-control}} \]
Design in the frequency domain
Compensate P-controller in frequency domain

Control structure starting with P-control:

\[ F(s) = \underbrace{\frac{K}{s}}_{P\text{-control}} F_{\text{lead}}(s) F_{\text{lag}}(s) \]

Lead-lag design

Shape the Bode plot/Nyquist curve of \( G_o \)

\[
\log |G_o(i\omega)| = \log |G(i\omega)| + \log K + \log |F_{\text{lead}}(i\omega)| + \log |F_{\text{lag}}(i\omega)|
\]

and

\[
\arg\{G_o(i\omega)\} = \arg\{G(i\omega)\} + 0 + \arg\{F_{\text{lead}}(i\omega)\} + \arg\{F_{\text{lag}}(i\omega)\},
\]

- \( F_{\text{lead}}(s) \) adjusts \( \omega_c \) (quickness) and \( \varphi_m \) (damping)
- \( F_{\text{lag}}(s) \) adjusts \( G_o(0) \) (accuracy)
Design in the frequency domain

Compensate P-controller in frequency domain

Control structure starting with P-control:

\[ F(s) = K \frac{F_{\text{lead}}(s)F_{\text{lag}}(s)}{F_{\text{P-control}}} \]

- Lead filter

\[ F_{\text{lead}}(s) = \frac{\tau_D s + 1}{\beta \tau_D s + 1}, \quad 0 \leq \beta < 1, \quad \tau_D > 0 \]

increases crossover \( \omega_c \) and phase margin \( \phi_m \)
Design in the frequency domain
Compensate P-controller in frequency domain

Control structure starting with P-control:

\[ F(s) = \underbrace{K}_{\text{P-control}} F_{\text{lead}}(s)F_{\text{lag}}(s) \]

- **Lead filter**
  \[ F_{\text{lead}}(s) = \frac{\tau_D s + 1}{\beta \tau_D s + 1}, \quad 0 \leq \beta < 1, \quad \tau_D > 0 \]
  increases phase margin \( \phi_m \) and crossover \( \omega_c \).

- **Lag filter**
  \[ F_{\text{lag}}(s) = \frac{\tau_I s + 1}{\tau_I s + \gamma}, \quad 0 \leq \gamma < 1, \quad \tau_I > 0 \]
  increases static gain \( G_c(0) \).

- **See ch. 5.4 for tuning principles**
Lead-lag design using Bode plot

Example of lead-lag controller

System $Y(s) = \frac{4}{(s+1)^2} U(s)$ with controller $U(s) = F(s)(R(s) - Y(s))$ where $F(s) = K$.

Performance specifications when $r(t)$ is a step:

- Accuracy, static error $e_f \leq 0.1$,
- Quickness, rise time $T_r \leq 0.5$,
- Damping, overshoot $M \leq 20\%$. 
Lead-lag design using Bode plot

Example of lead-lag controller

Bode plots for

- system $G(s)$,
- open-loop $G_o(s) = F(s)G(s)$ with (stable) P-controller
- add lead-lag in $F(s)$ to compensate
Lead-lag design using Bode plot

Example of lead-lag controller

Step response $y(t)$ using lead-lag compensation to P-controller:

$$F(s) = K F_{\text{lead}}(s) F_{\text{lag}}(s).$$

Performance specifications are met with lead-lag controller:

- $T_r = 0.45$ seconds, $M = 8\%$, $e_f < 0.1$. 
Minimum phase systems
Several different systems $G(s)$ may have identical magnitude curve $|G(i\omega)|$. Only phase curve $\text{arg}\{G(i\omega)\}$ will differ.

Definition:
Among all systems $G(s)$ with the same magnitude curve, the system with the least negative phase shift is a minimum phase system.
Minimum phase systems

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Among all systems $G(s)$ with the same magnitude curve, the system with the least negative phase shift is a minimum phase system.

**Result 5.1**
A system is minimum phase $\iff$ it has no poles nor zeros in the right half-plane, and contains no time delays.
Minimum phase systems

Several different systems $G(s)$ may have identical magnitude curve $|G(i\omega)|$. Only phase curve $\arg\{G(i\omega)\}$ will differ.

**Definition:**
Among all systems $G(s)$ with the *same magnitude curve*, the system with the *least* negative phase shift is a *minimum phase system*.

**Result 5.1**
A system is *minimum phase* $\iff$ it has no poles nor zeros in the right half-plane, and contains no time delays.

Cf. bicycle as non-minimum phase!
Non-minimum phase systems

Example

Exemple of open-loop systems

- \( G_o(s) = \frac{s+1}{s(s^2+2s+2)} \): \( p_i = 0, -1 \pm i \). \( z_i = -1 \)
- \( G_o'(s) = \frac{-s+1}{s+1} \cdot G_o(s) \): \( p_i = 0, -1 \pm i \). \( z_i = +1 \)
- \( G_o''(s) = e^{-2s}G_o(s) \): as above but time-delay 2 sec.
Non-minimum phase systems

Example

Exemple of open-loop systems

\[ G_o(s) = \frac{s+1}{s(s^2+2s+2)} : p_i = 0, -1 \pm i. \ z_i = -1 \]

\[ G'_{o}(s) = -s + 1 + \frac{s}{s+1} \cdot G_{o}(s) : p_i = 0, -1 \pm i. \ z_i = +1 \]

\[ G''_{o}(s) = e^{-2s} G_{o}(s) : as \ above \ but \ time-delay \ 2 \ sec. \]

Bode plots for \( G_{o}, G'_{o} \ & G''_{o} \):
Non-minimum phase systems

Examples

Magnitude curves for corresponding closed-loop systems, $G_c$, $G'_c$ & $G''_c$:

Non-minimum phase systems are hard to control!
Summary and recap

- Frequency response and Bode plots
- Performance metrics in the frequency domain
  - Bandwidth
  - Resonance peak
  - Static gain
- Open-loop design and Nyquist curve
- Minimum phase systems