



# Intro. Computer Control Systems: F7

## State-space descriptions of systems

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# F6: Quiz!

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- 1) The **bandwidth**  $\omega_B$  of the closed-loop system  $G_c(s)$  affects its
  - a quickness  $\uparrow$
  - b damping  $\uparrow$
  - c stability  $\downarrow$

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  - a quickness  $\uparrow$
  - b damping  $\uparrow$
  - c stability  $\downarrow$
  
- 2) A **non-minimum phase system**
  - a is always unstable  $\uparrow$
  - b may have a zero in the right half-plane  $\uparrow$
  - c is easy to control  $\downarrow$



## Systems in state-space description

# Linear time-invariant systems



Different mathematical forms of *same model*:

1. ODE:

$$\frac{d^n}{dt^n}y + \dots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \dots + b_{m-1}\frac{d}{dt}u + b_m u$$

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2. Transfer function:

$$Y(s) = G(s)U(s) \quad \text{ignoring initial conditions}$$

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2. Transfer function:

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3. State-space description:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

# System states

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- ▶ Alternative description using  $n$  state variables

$$y(t) = \sum_{i=1}^n c_i \underbrace{x_i(t)}_{\text{state variables}} + Du(t) = Cx(t) + Du(t)$$



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$$y(t) = \sum_{i=1}^n c_i \underbrace{x_i(t)}_{\text{state variables}} + Du(t) = C\mathbf{x}(t) + Du(t)$$

where

$$C = [c_1 \quad \cdots \quad c_n] \quad \text{and} \quad \mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$



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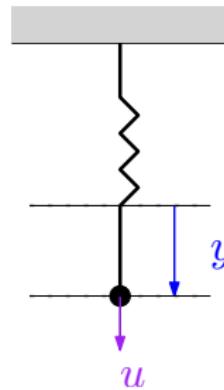
Note: matrix multiplication and eigenvalues necessary!



## Building intuition

# Build intuition from simple systems

Example: State vector in space  $\mathbb{R}^2$



Figur: Force  $u(t)$  and position  $y(t)$ .

Standard form of linear ODE model:

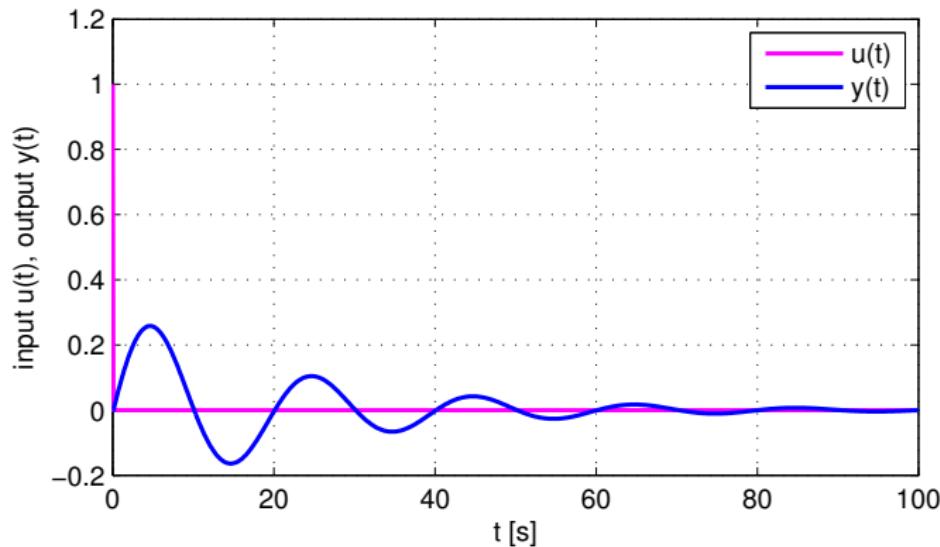
$$\frac{d^2}{dt^2} \mathbf{y} + \left( \frac{K}{m} \right) \mathbf{y} = \left( \frac{1}{m} \right) \mathbf{u}$$

[Board: derive state-space description]

# Build intuition from simple systems

Example: State vector in space  $\mathbb{R}^2$

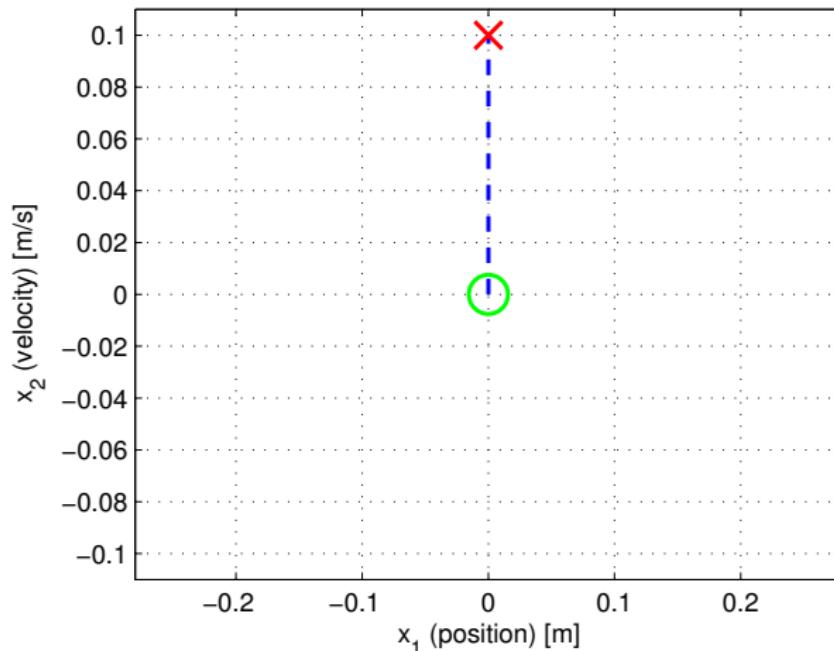
Input  $u(t)$  impulse



Output  $y(t) = Cx(t)$

# Build intuition from simple systems

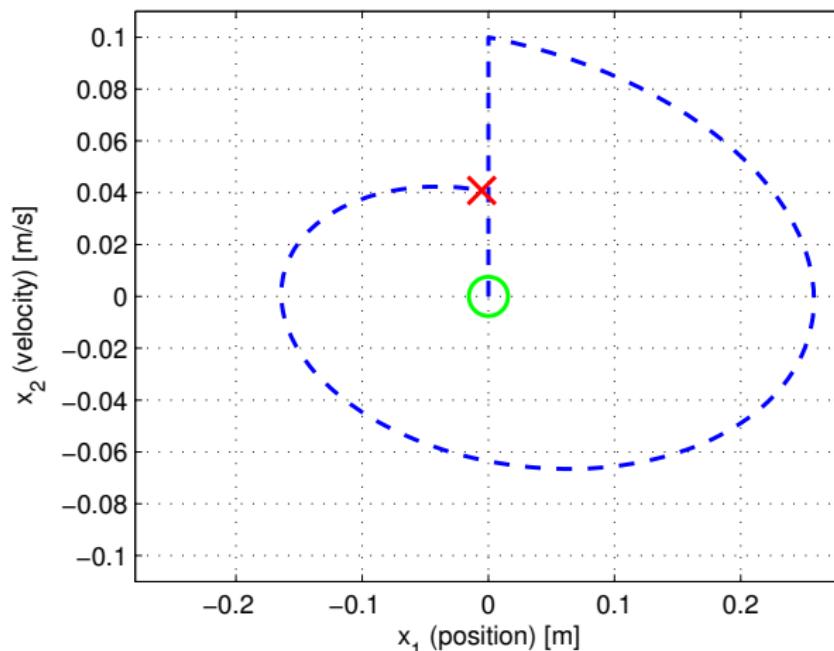
Example: State vector in space  $\mathbb{R}^2$



$x(t)$  at  $t = 0^+$

# Build intuition from simple systems

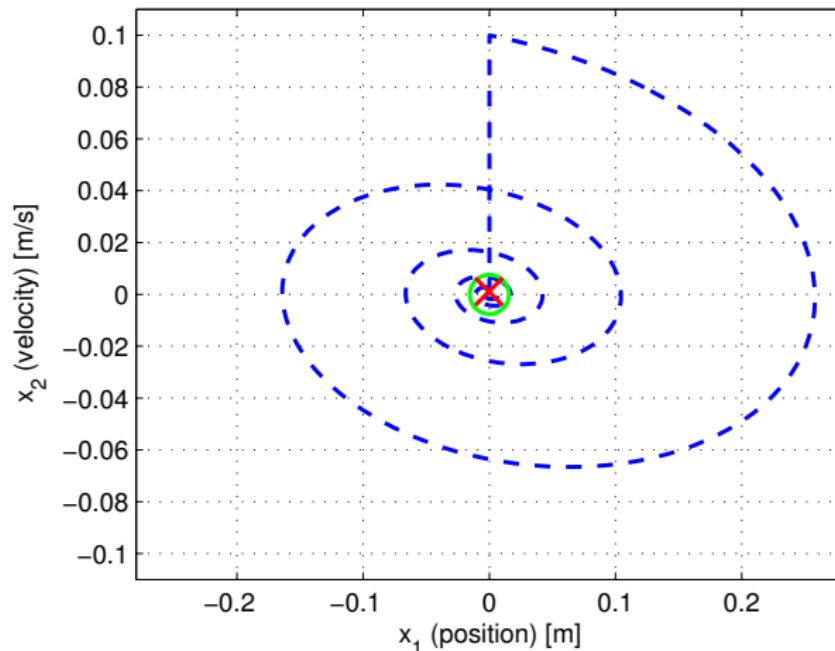
Example: State vector in space  $\mathbb{R}^2$



$x(t)$  at  $t = 20$

# Build intuition from simple systems

Example: State vector in space  $\mathbb{R}^2$

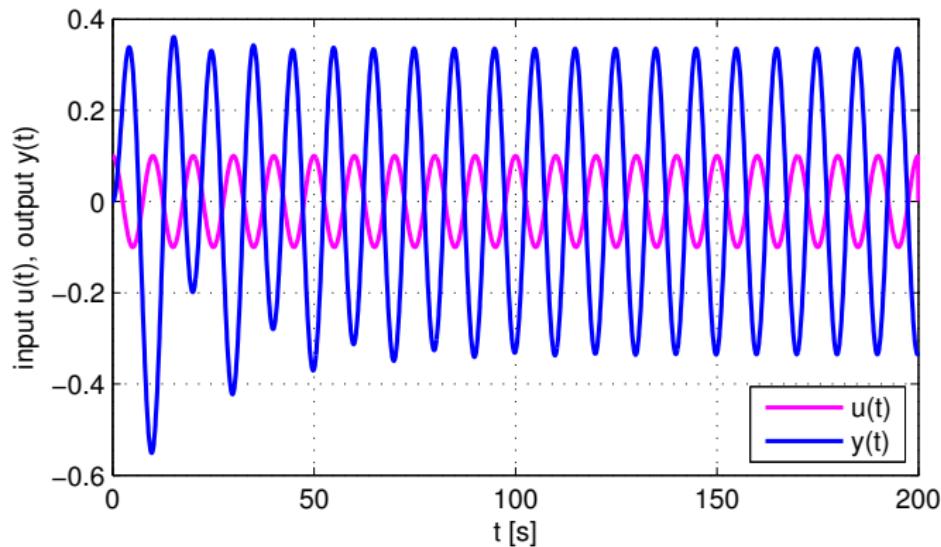


$x(t)$  at  $t = 100$

# Build intuition from simple systems

Example: State vector in space  $\mathbb{R}^2$

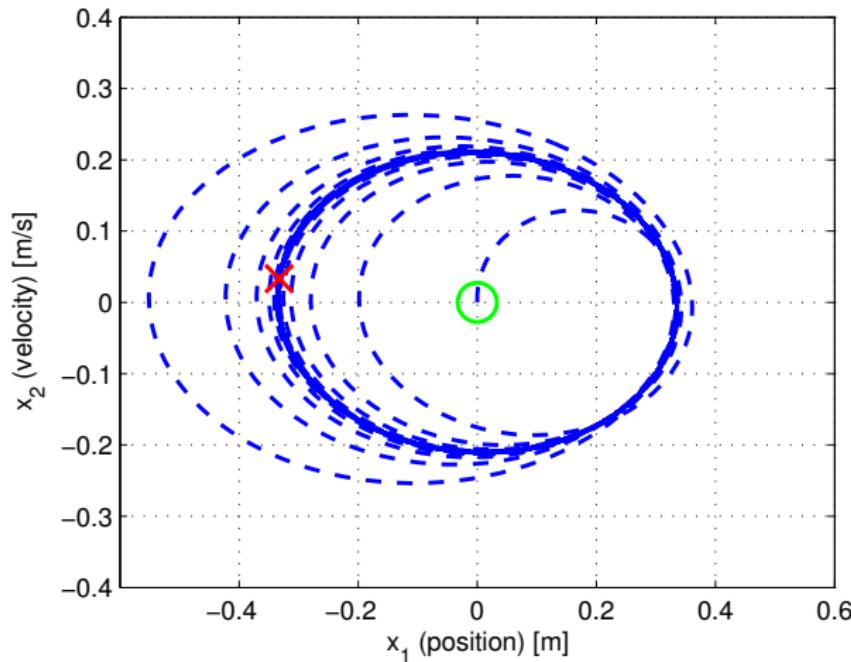
Input  $u(t)$  sine



Output  $y(t) = Cx(t)$

# Build intuition from simple systems

Example: State vector in space  $\mathbb{R}^2$



$x(t)$  at  $t = 200$

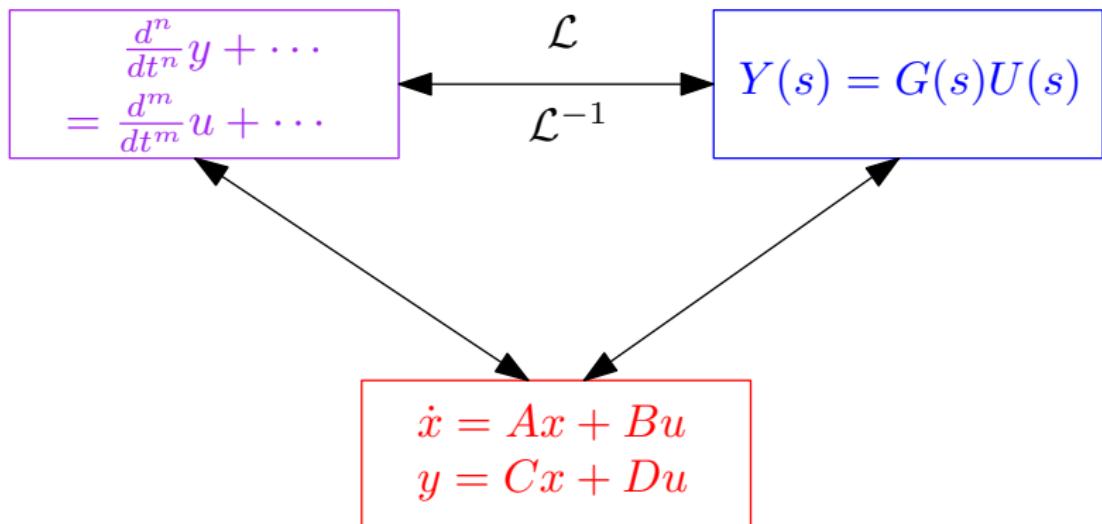


## Relation between system descriptions

# Linear time-invariant system models

## Relations between mathematical descriptions

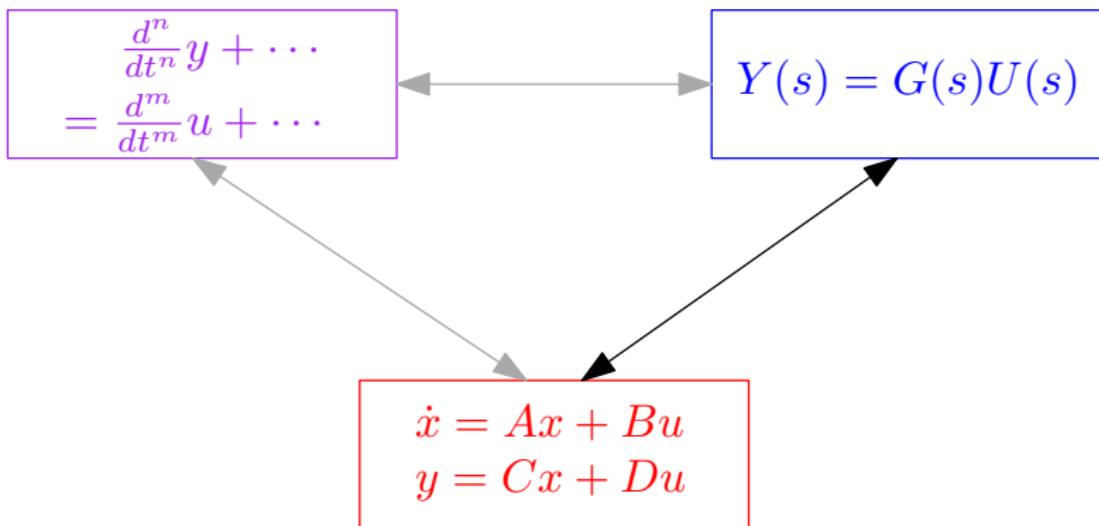
Descriptions with different strengths



# Linear time-invariant system models

## Relations between mathematical descriptions

Translate from one description to the next



# Relations between descriptions

State-space form → transfer function



[Board: Laplace transform and solve  $G(s)$ ]

# Relations between descriptions

State-space form → transfer function

$$\begin{array}{|c|} \hline \dot{x} = Ax + Bu \\ y = Cx + Du \\ \hline \end{array} \longrightarrow \boxed{Y(s) = G(s)U(s)}$$

Transfer function obtained by

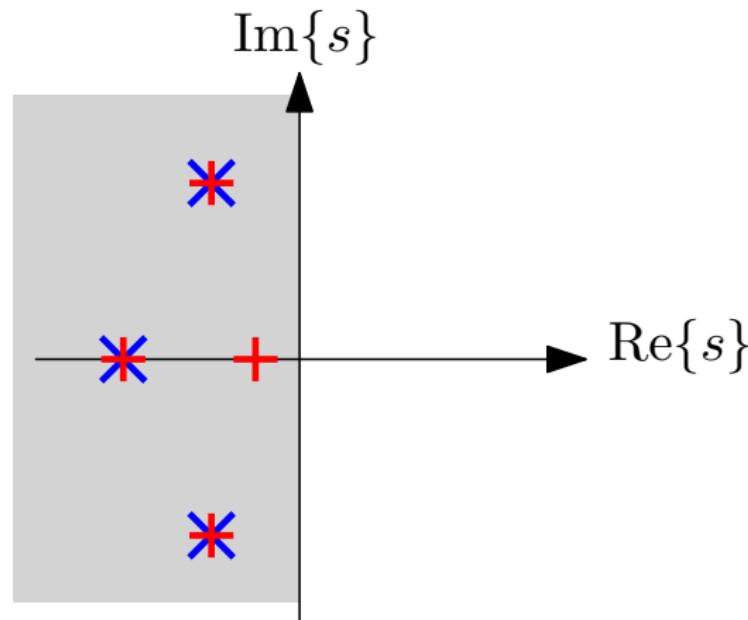
$$G(s) = \underbrace{C}_{1 \times n} \underbrace{(sI - A)^{-1}}_{n \times n} \underbrace{B}_{n \times 1} + \underbrace{D}_{1 \times 1} = \frac{b(s)}{a(s)}$$

Important property:

- ▶ System matrix  $A$ :s eigenvalues  $\{\lambda_i\}$  given by solution to  $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$
- ▶  $a(s) = \det(sI - A)$  is a polynomial of order  $n$

# Relations between descriptions

State-space form → transfer function



$G(s)$ :s poles  $p_i \subseteq A$ :s eigenvalues  $\lambda_j$

# Relations between descriptions

Transfer function → State-space form



Choice of state variables and system matrices **not unique!**

[Board: alternative states  $z = Tx$ ]

# Relations between descriptions

Transfer function → State-space form

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

one can choose e.g. controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = [b_1 - a_1 b_0 \quad b_2 - a_2 b_0 \quad \cdots \quad b_n - a_n b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

# Relations between descriptions

Transfer function → State-space form

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

one can choose e.g. observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

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# Relations between descriptions

Transfer function → State-space form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

$$Y(s) = G(s)U(s)$$

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

we have general state-space form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

**Note:** when order of numerator  $< n$  we obtain  $D = 0$



## Trajectory of states

# Solution to state-space equation

## First-order system

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Output  $y(t) = cx(t) + du(t)$  where  $x(t)$  given by solution to

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$$X(s) = \frac{1}{s-a}x_0 + \frac{b}{s-a}U(s)$$

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$$X(s) = \frac{1}{s-a}x_0 + \frac{b}{s-a}U(s)$$

and inverse transform yields solution

$$x(t) = e^{at}x_0 + \int_0^t \underbrace{e^{a\tau} b}_{h(\tau)} u(t-\tau) d\tau$$

# Exponentials and matrix exponentials

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- Exponential  $e^{at}$  is a function which fulfills

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# Solution to state-space equation

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[Board: Laplace + inverse transform]

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and inverse transform gives solution

$$x(t) = e^{At}x_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau$$

Note: matrix exponential  $e^{At}$



## Stable states and stable systems

# Stability

The state evolve according to:

$$\mathbf{x}(t) = e^{At} \mathbf{x}_0 + \int_0^t e^{A\tau} B u(t - \tau) d\tau$$

Asymptotically stable if

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = 0 \quad \text{when } u(t) \equiv 0$$

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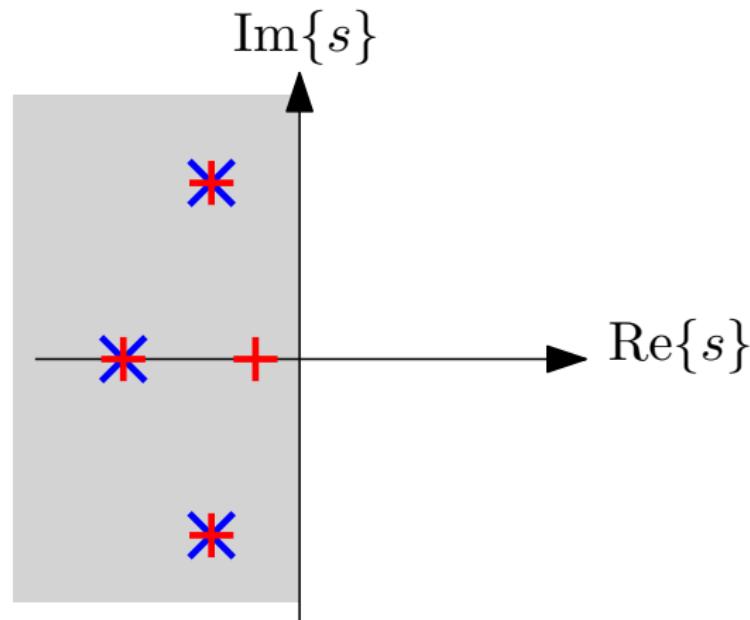
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## Input-output stability of system

$A$ :s eigenvalues are strictly in left half-plane  $\Rightarrow$  system is input-output stable

# Stability



$G(s)$ :s poles  $p_i \subseteq A$ :s eigenvalues  $\lambda_j$

# Summary and recap

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- ▶ State-space description using vectors and matrices
- ▶ System matrices and transfer functions
- ▶ Solution to state-space equation
- ▶ Stability concepts