



Intro. Computer Control Systems: F8

Properties of state-space descriptions and feedback

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F7: Quiz!

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- 1) The state-space description of a system is
- a not unique ↑
 - b unique ↑
 - c stable ↓

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- 2) The eigenvalues of the system matrix A reveals something about
 - a poles ↑
 - b zeros ↑
 - c the closed-loop system ↓



F7: Quiz!

-
- 1) The state-space description of a system is
 - a not unique ↑
 - b unique ↑
 - c stable ↓
 - 2) The eigenvalues of the system matrix A reveals something about
 - a poles ↑
 - b zeros ↑
 - c the closed-loop system ↓
 - 3) Solution to $\dot{x} = Ax + Bu$ with initial condition x_0 is obtained using
 - a a linear system of equations ↑
 - b the matrix exponential ↑
 - c the Nyquist curve ↓



Nonlinear time-invariant systems

Nonlinear systems and states

Most systems are nonlinear!



Nonlinear differential equations:

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

Linearize around *operating point* x_0, u_0 . Typically use a **stationary point**: $\dot{x} = f(x_0, u_0) = 0$



Nonlinear systems and states

Nonlinear differential equations:

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u)\end{aligned}$$

Taylor series expansion around *stationary point* x_0, u_0 with $y_0 = h(x_0, u_0)$ results in **linear deviation model**:

$$\boxed{\begin{aligned}\dot{\bar{x}} &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} + D\bar{u}\end{aligned}}$$

Nonlinear systems and states

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- ▶ Linear state-space description of the deviations $\bar{x} = x - x_0$ **around** the operating point of system x_0 .
- ▶ Matrices A, B, C and D given by **derivatives** of $f(x, u)$ and $h(x, u)$ with respect to x and u . **See ch. 8.4 G&L.**



Feedback control using states

State-feedback control

State space description of linear time-invariant system

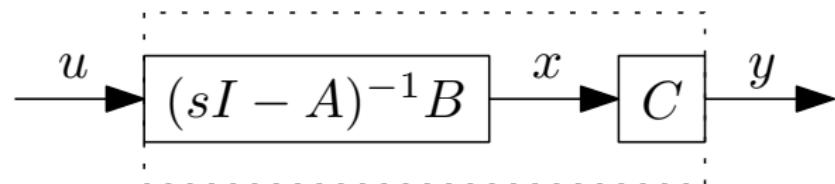
$$\begin{aligned} \dot{x} = Ax + Bu \\ y = Cx \end{aligned} \quad \Leftrightarrow \quad Y(s) = G(s)U(s)$$



State-feedback control

State space description of linear time-invariant system

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\quad \Rightarrow \quad G(s) = C(sI - A)^{-1}B$$

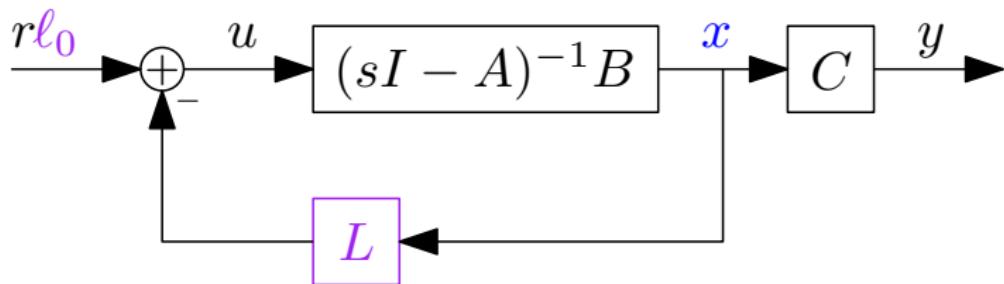


State-feedback control

Idea: Feedback control using states

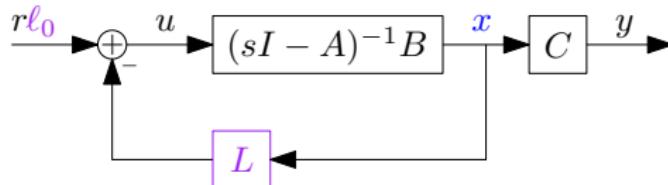
$$u = -Lx + \ell_0 r,$$

where L and ℓ_0 are design parameters.



$$\Rightarrow \dot{x} = Ax + B \underbrace{(-Lx + \ell_0 r)}_{=u}$$

State-feedback control



Closed-loop system from r to y becomes:

$$\begin{aligned}\dot{x} &= Ax + B(-Lx + l_0r) = (A - BL)x + Bl_0r \\ y &= Cx\end{aligned}$$

Is it possible to

- ▶ control the system to *all* states x^* in \mathbb{R}^n ?
- ▶ design the *closed-loop system's* poles?
- ▶ (estimate the state $x(t)$?)

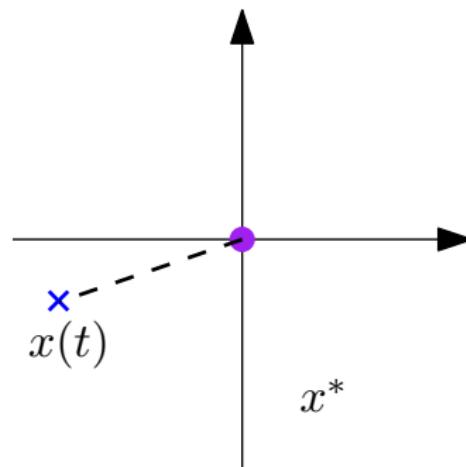


Controlling the states?

Controllability

Can we reach all states?

A sought state x^* is **controllable** if some input $u(t)$ can move the system from $x(0) = 0$ to $x(T) = x^*$

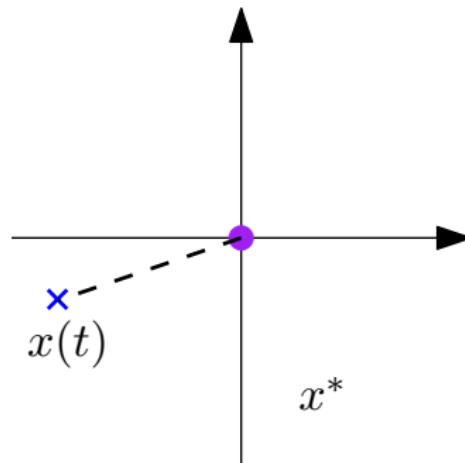


Controllability

Can we reach all states?

When $x_0 = 0$, we obtain the state at $t = T$ as:

$$x(T) = e^{At}x_0 + \int_0^T e^{A\tau} Bu(T - \tau)d\tau$$

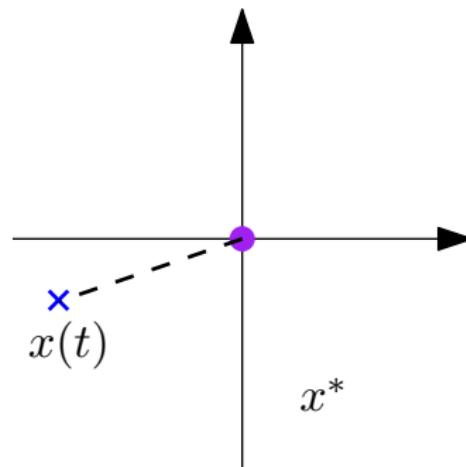


Controllability

Can we reach all states?

When $x_0 = 0$, we obtain the state at $t = T$ as:

$$\begin{aligned}x(T) &= 0 + \int_0^T e^{A\tau} Bu(T - \tau)d\tau \\&= [\text{via Cayley-Hamiltons theorem}] \\&= B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}\end{aligned}$$



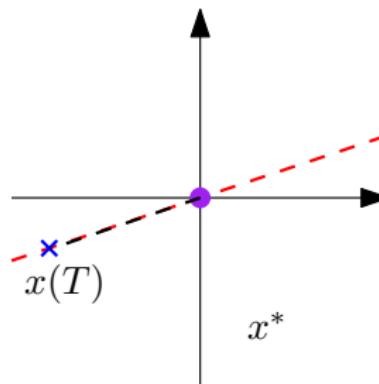
Controllability

Can we reach all states?

When $x_0 = 0$, we obtain the state at $t = T$ as:

$$x(T) = B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}$$

is a linear combination of $B, AB, \dots, A^{n-1}B$.

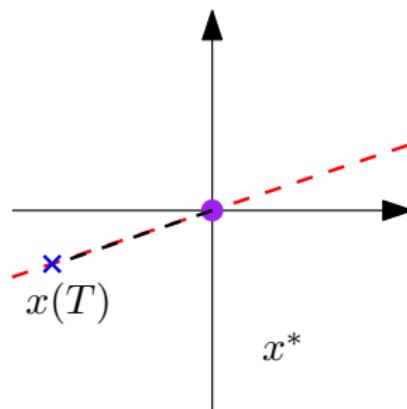


A state x^* is **controllable** if it can be expressed as such a linear combination, i.e., if x^* is in the **column space** of

$$\mathcal{S} \triangleq [B \ AB \ \cdots \ A^{n-1}B]$$

Controllability

Can we reach all states?



Figur : Example column space of \mathcal{S} and non-controllable state x^* .

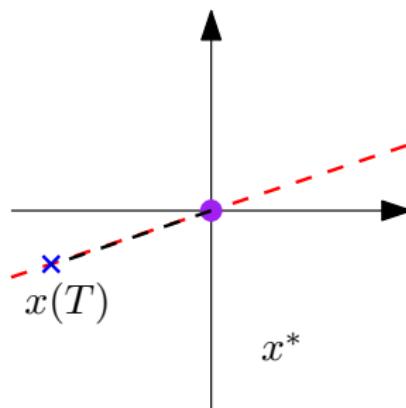
Controllable system

All states x^* are controllable $\Leftrightarrow \mathcal{S}$:s columns are linearly independent

Note: $\text{rank}(\mathcal{S}) = n$ or $\det(\mathcal{S}) \neq 0$

Controllability

Can we reach all states?



Figur : Example column space of \mathcal{S} and non-controllable state x^* .

Controllable canonical form

System is controllable \Leftrightarrow It can be written on controllable canonical form

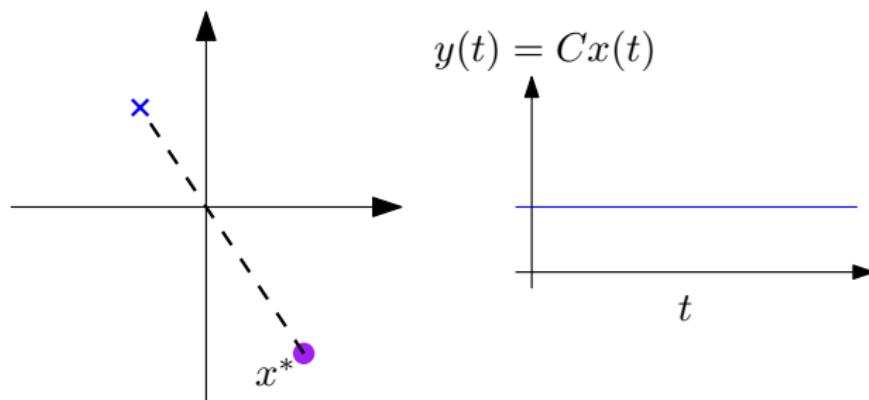


Observing the states?

Observability

Can we observe all states through output?

Assume $u(t) \equiv 0$.



A state $x^* \neq 0$ is **unobservable** if the output $y(t) \equiv 0$ when system starts at $x(0) = x^*$.

Observability

Can we observe all states through output?

When $u(t) \equiv 0$ we obtain

$$\begin{aligned}y(t) &= Cx(t) \\&= Ce^{At}x^* + 0\end{aligned}$$

When $y(t) \equiv 0$ we do not observe any changes in the output:

$$\frac{d^k}{dt^k}y(t)\Big|_{t=0} = CA^kx^* = 0.$$

That is,

$$Cx^* = 0, \quad CAx^* = 0, \quad \dots, \quad CA^{n-1}x^* = 0$$

Observability

Can we observe all states through output?

When $u(t) \equiv 0$ and $y(t) \equiv 0$ we observe no changes:

$$Cx^* = 0, \quad CAx^* = 0, \quad \dots, \quad CA^{n-1}x^* = 0$$

or

$$\mathcal{O}x^* = 0$$

where

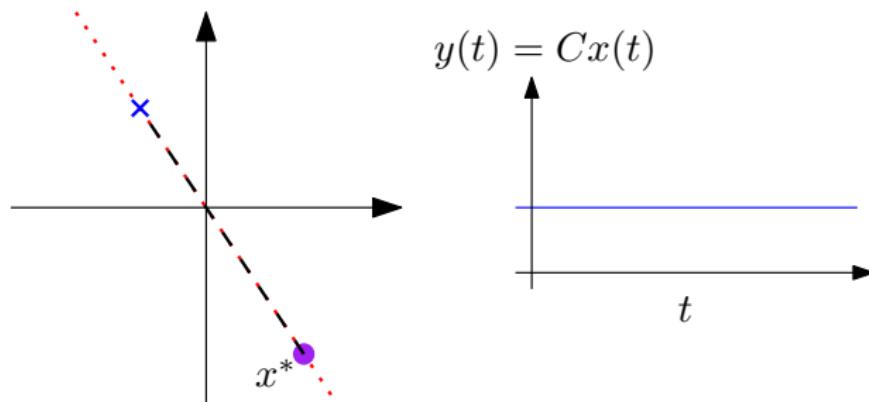
$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Therefore:

- ▶ A state $x^* \neq 0$ is **unobservable** if it belongs to the null space of \mathcal{O} .

Observability

Can we observe all states through output?



Figur : Example null space of \mathcal{O} and unobservable state x^* .

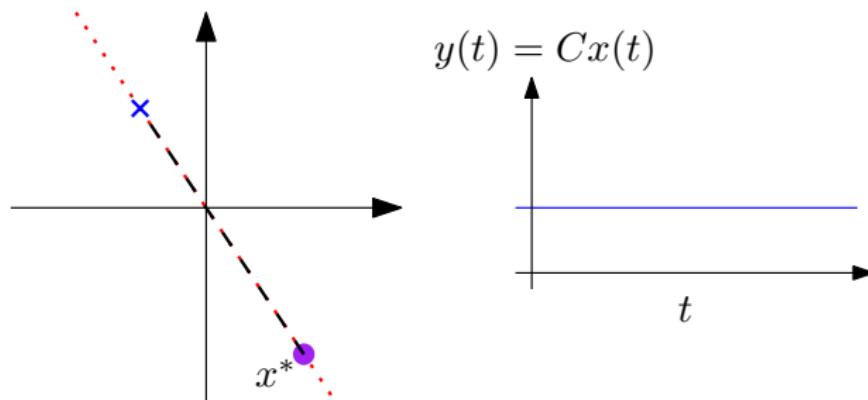
Observable system

All states x^* are observable $\Leftrightarrow \mathcal{O}$:s columns are linearly independent

Note: $\text{rank}(\mathcal{O}) = n$ or $\det(\mathcal{O}) \neq 0$

Observability

Can we observe all states through output?



Figur : Example null space of \mathcal{O} and unobservable state x^* .

Observable canonical form

System is observable \Leftrightarrow It can be written on observable canonical form



Build intuition

Build intuition from simple systems

Example: controllable system

System on controllable canonical form:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= [1 \quad 1] x(t)\end{aligned}$$

Transfer function:

$$G(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2 + 2s + 1} = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$$

[Board: investigate observability using \mathcal{O}]

Build intuition from simple systems

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[Board: investigate observability using \mathcal{O}]

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \det \mathcal{O} = 0 \Leftrightarrow \text{unobservable}$$

Build intuition from simple systems

Example: observable system

System on observable canonical form:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= [1 \ 0] x(t)\end{aligned}$$

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Build intuition from simple systems

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[Board: investigate controllability using \mathcal{S}]

$$\mathcal{S} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \det \mathcal{S} = 0 \quad \Leftrightarrow \quad \text{non-controllable}$$

Build intuition from simple systems

Exemple: controllable and observable system

Systems in previous examples have the same transfer function

$$G(s) = \frac{1}{s + 1}.$$

Can also be written in state-space form

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t), \\ y(t) &= x(t).\end{aligned}$$

where $x(t)$ is a scalar.

[Board: investigate \mathcal{S} and \mathcal{O}]

Build intuition from simple systems

Exemple: controllable and observable system

Systems in previous examples have the same transfer function

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where $x(t)$ is a scalar.

[Board: investigate \mathcal{S} and \mathcal{O}]

$$\begin{array}{lll}\mathcal{S} = 1 & \Rightarrow \det \mathcal{S} = 1 & \\ \mathcal{O} = 1 & \Rightarrow \det \mathcal{O} = 1 & \end{array} \Leftrightarrow \text{controllable and observable} \quad (1)$$

Note: we eliminated “invisible states”



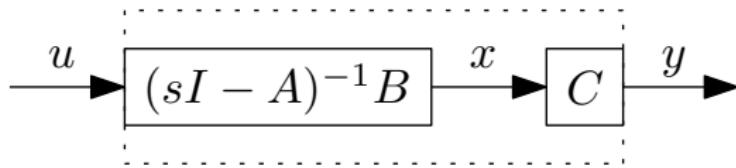
Minimal realization

Minimal realization

System with transfer function $G(s)$ and state-space form

$$\dot{x} = Ax + Bu$$

$$y = Cx$$



Definition 8.2 G&L

State-space form of $G(s)$ is a **minimal realization** if vector x has the smallest possible dimension.

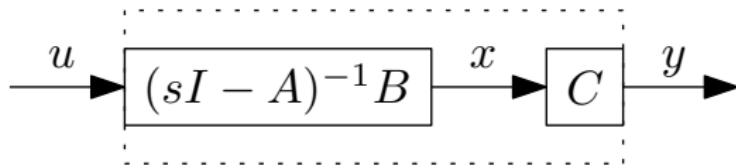


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Definition 8.2 G&L

State-space form of $G(s)$ is a **minimal realization** if vector x has the smallest possible dimension.

Result 8.11(+8.12) G&L

A state-space form is **minimal realization** \Leftrightarrow controllable and observable $\Leftrightarrow A$:s eigenvalues = $G(s)$:s poles



Design of state-feedback control

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \textcolor{blue}{\ell}_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \ell_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

Closed-loop system

$$\begin{aligned}\dot{x} &= (A - B\textcolor{blue}{L})x + B\ell_0 r \\ y &= Cx\end{aligned}$$

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \ell_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

Closed-loop system as a transfer function

Output is $Y(s) = \textcolor{pink}{G}_c(\textcolor{violet}{s})R(s)$, where

$$\textcolor{pink}{G}_c(s) = C(sI - A + B\textcolor{blue}{L})^{-1}B\ell_0$$

State-feedback control

State-space model with controller $u = -\textcolor{blue}{L}x + \ell_0 r$ where

$$\textcolor{blue}{L} = [\ell_1 \quad \ell_2 \quad \cdots \quad \ell_n]$$

System matrix of closed-loop system:

$$(A - B\textcolor{blue}{L})$$

Eigenvalues/**poles** given by polynomial equation

$$\det(sI - A + B\textcolor{blue}{L}) = 0$$

which we can *design* via $\textcolor{blue}{L}$!

State-feedback control

Design of the gain ℓ_0

- ▶ $Y(s) = \textcolor{red}{G}_c(s)R(s)$ where

$$\textcolor{red}{G}_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

- ▶ It is *desirable* to have at least $\textcolor{red}{G}_c(0) = 1$

State-feedback control

Design of the gain ℓ_0

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$$\textcolor{red}{G}_c(s) = C(sI - A + \textcolor{blue}{B}\textcolor{blue}{L})^{-1}B\ell_0.$$

- ▶ It is *desirable* to have at least $\textcolor{red}{G}_c(0) = 1$
- ▶ $\textcolor{red}{G}_c(0) = C(-A + \textcolor{blue}{B}\textcolor{blue}{L})^{-1}B\ell_0 = 1$ and so

$$\boxed{\ell_0 = \frac{1}{C(-A + \textcolor{blue}{B}\textcolor{blue}{L})^{-1}B}}$$

State-feedback control

Design of the gain ℓ_0

- $Y(s) = \mathbf{G}_c(s)R(s)$ where

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- $\mathbf{G}_c(0) = C(-A + B\mathbf{L})^{-1}B\ell_0 = 1$ and so

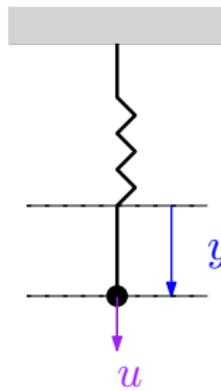
$$\boxed{\ell_0 = \frac{1}{C(-A + B\mathbf{L})^{-1}B}}$$

- (More generally, replace $\ell_0 r$ with $\mathbf{F}_r(s)R(s)$)

How to design \mathbf{L} ?

Build intuition from simple systems

Exemple: state-vector in \mathbb{R}^2



Figur : Force $u(t)$ and position $y(t)$.

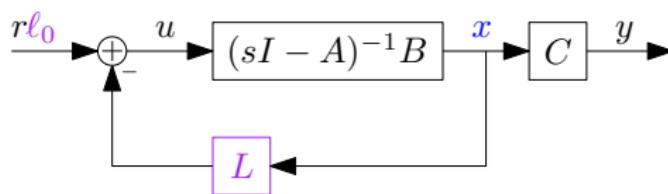
State-space form:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u \\ y &= [1 \quad 0] x\end{aligned}$$

[Board: design L so that closed-loop system has poles -2 and -3]

Pole placement

State-feedback control

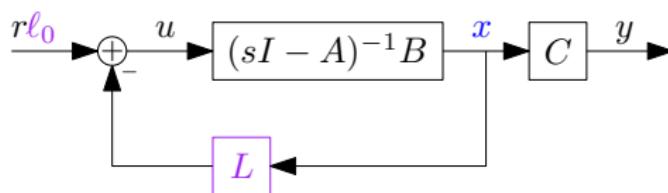


Result 9.1

State-space form is **controllable** $\Leftrightarrow L$ can be designed to yield **arbitrarily placed poles** (real and complex-conjugated) of the closed-loop system

Pole placement

State-feedback control



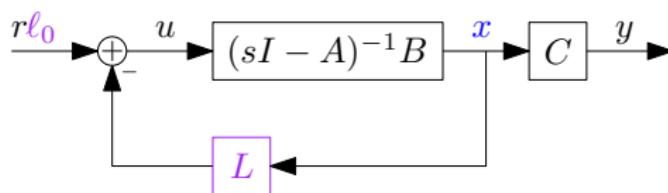
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- ▶ L solved by $\det(sI - A + BL) = 0$ with **desired roots**
- ▶ L **very simple** to solve for system on controllable canonical form

Pole placement

State-feedback control



Result 9.1

State-space form is **controllable** $\Leftrightarrow L$ can be designed to yield **arbitrarily placed poles** (real and complex-conjugated) of the closed-loop system

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- ▶ L **very simple** to solve for system on controllable canonical form

What to do when we **can't** measure x directly?

Summary and recap

- ▶ Linearization of nonlinear system models
- ▶ Properties:
 - ▶ Controllable
 - ▶ Observable
 - ▶ Minimal realization
- ▶ State-feedback control
- ▶ Pole placement for the closed-loop system