
Sensitivity and robustness

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F9: Quiz!

1) When a system is observable
   a) the states can be estimated arbitrarily well
   b) the states can be controlled arbitrarily well
   c) the system is also stable

2) State estimation using an observer
   a) does not handle initial errors of the state
   b) can be described as a differential equation
   c) is an unstable process

3) The transfer function for a control system with estimated states
   a) is different from that of control system with known states
   b) is the same as that of control system with known states
   c) is real-valued

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   b is the same as that of control system with known states↑
   c is real-valued ↓
Sensitivity to disturbance and noise
Control system with disturbances and noise
Using general linear feedback

Closed-loop system using general linear feedback:

\[ G_c(s) = \frac{G(s)F_r(s)}{1 + G(s)F_y(s)} \]

General open-loop system: \( G_o(s) \triangleq F_y(s)G(s) \)
How will the control system cope with unknown disturbances and noise?

[Board: the closed-loop system with $V(s)$ and $N(s)$]
Defining sensitivity functions

- **Sensitivity function:**

\[
S(s) \triangleq \frac{1}{1 + G_o(s)}
\]

- **Complementary sensitivity function:**

\[
T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)}
\]

- **Consequence:**

\[
S(s) + T(s) \equiv 1, \quad \forall s
\]

and \(S(s)\) and \(T(s)\) affected by controller \(F_y(s)\).
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- **S(s) and T(s) affected by controller** \( F_y(s) \).
Closed-loop system and the sensitivity functions

- Closed-loop system:
  \[ Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s) \]
Closed-loop system and the sensitivity functions

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- Want both \(|S(i\omega)|\) and \(|T(i\omega)| \ll 1\) simultaneously...
Closed-loop system and the sensitivity functions

- Closed-loop system:
  \[ Y(s) = G_c(s)R(s) + S(s)V(s) - T(s)N(s) \]

- Want both \(|S(i\omega)|\) and \(|T(i\omega)| \ll 1\) simultaneously...
- ...but impossible since
  \[ |S(i\omega)| + |T(i\omega)| \geq |S(i\omega) + T(i\omega)| \equiv 1 \]
Sensitivity functions in frequency domain

Design trade-off

Example:

- **Disturbance** $v(t)$ with energy at low frequencies
- **Noise** $n(t)$ with energy at high frequencies
Sensitivity functions in frequency domain

Design trade-off

Example:
- Disturbance $v(t)$ with energy at low frequencies
- Noise $n(t)$ with energy at high frequencies

Typical design trade-off is then:
- low $\omega$: $|S(i\omega)| \ll 1$ to suppress $V(i\omega)$.
- high $\omega$: $|T(i\omega)| \ll 1$ to suppress $N(i\omega)$. 
In addition we want Nyquist contour

$$G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)}$$

far from $-1$. (Cf. F6 and F7.)
Design of poles and zeros via $F_r$ and $F_y$ affects also $S$ and $T$

E.g. quick system means also increased sensitivity
Robustness to model errors
Control systems with model errors

Model $G$ is an approximation

Assume that the real system can be written as

$G_0(s) = G(s)(1 + \Delta G(s))$

The relative model error of $G(s)$:

$\Delta G(s) = G_0(s) - \frac{G_0(s)}{G(s)}$

How is stability of $G_0(s)$ affected by unknown error $\Delta G(s)$?
Control systems with model errors

Model $G$ is an approximation

Assume that the real system can be written as

$$G^0(s) = G(s)(1 + \Delta G(s))$$

The relative model error of $G(s)$:

$$\Delta_G(s) = \frac{G^0(s) - G(s)}{G(s)}$$
Control systems with model errors

Assume that the real system can be written as

\[ G^0(s) = G(s)(1 + \Delta_G(s)) \]

The relative model error of \( G(s) \):\[
\Delta_G(s) = \frac{G^0(s) - G(s)}{G(s)}
\]

How is stability of \( G^0_c(s) \) affected by unknown error \( \Delta_G(s) \)?
Model errors and stability
Using the complementary sensitivity function

Assume:

1. Controller $F(s)$ stabilizes the assumed system $G(s)$
2. $G(s)$ and $G^0(s)$ have same number of poles in right half-plane.
3. Open-loop: $F(s)G(s) \to 0$ and $F(s)G^0(s) \to 0$ where $|s| \to \infty$
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(Result 6.2) Robustness criterion

If assumptions are valid and $T(i\omega)$ fulfills

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad -\infty \leq \omega \leq \infty$$

$\Rightarrow$ the real closed-loop system $G^0_c(s)$ is also stable!
Model errors and stability

Bounding the model errors

$\Delta G(i\omega)$ is unknown but suppose we can cap it by $g(\omega) > |\Delta G(i\omega)|$
Model errors and stability

Bounding the model errors

\( \Delta_G(i\omega) \) is unknown but suppose we can cap it by \( g(\omega) > |\Delta_G(i\omega)| \)

\[ |T(i\omega)| < \frac{1}{g(\omega)} < \frac{1}{|\Delta_G(i\omega)|} \]

\( \Rightarrow \) real closed-loop system \( G_c^0(s) \) is also stable
Summary and recap

- Sensitivity with respect to disturbances and noise
- Sensitivity functions and their impact on control
- Robustness with respect to model errors
- Robustness criterion in the frequency domain