Intro. Computer Control Systems: 
F2
Transfer function, poles and stability

Dave Zachariah

Dept. Information Technology, Div. Systems and Control
F1: Quiz!

Suppose a control system

\[ r(t) \quad \text{Controller} \quad u(t) \quad \text{System} \quad y(t) \]

comes with two different settings (a) and (b).
1) Which setting of the controller is intuitively better?
   a. Setting to the left ↑
   b. Setting to the right ↑
   c. They are equally good ↓
Linear system models
Linear time-invariant models are useful and sufficiently accurate in many control applications.

\[
u(t) \quad \rightarrow \quad G \quad \rightarrow \quad y(t)
\]

Linear ODE:s are one possible input-output description, i.e. of \( G \):

\[
\frac{d^n}{dt^n} y + \cdots + a_{n-1} \frac{d}{dt} y + a_n y = b_0 \frac{d^m}{dt^m} u + \cdots + b_{m-1} \frac{d}{dt} u + b_m u
\]

with initial conditions.
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with initial conditions.

Rarely practical in analysis or design for control!
Laplace transform

Used as tool to solve and analyze linear ODE:s

- **Notation:**

\[
\begin{align*}
  y(t) & \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \mathcal{L} [y(t)] = Y(s)
\end{align*}
\]
Laplace transform

Used as tool to solve and analyze linear ODE:s

- **Notation:**
  \[ y(t) \leftrightarrow \mathcal{L}[y(t)] = Y(s) \]

- **Definition:**
  \[ Y(s) = \mathcal{L}[y(t)] = \int_{0}^{\infty} y(t)e^{-st}dt, \quad s \in \mathbb{C} \]

Inverse transform:
\[ y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi i} \int_{\mathbb{C}} Y(s)e^{st}ds, \quad s \in \mathbb{C} \]

Note that \( s \) and \( Y(s) \) are complex-valued!
Important properties

linearity: \[ y(t) = \alpha x(t) + \beta z(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad Y(s) = \alpha X(s) + \beta Z(s) \]

derivatives:
\[
\begin{align*}
\frac{dy}{dt} & \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad sY(s) - y(0) \\
\frac{d^2 y}{dt^2} & \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad s^2 Y(s) - sy(0) - \dot{y}(0) \\
\vdots & \\
\int_0^t y(\tau) d\tau & \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \frac{1}{s} Y(s) \\
\int_0^t x(\tau) z(t - \tau) d\tau & \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad X(s) Z(s) \\
\end{align*}
\]

final-value thm.*:
\[
\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s)
\]
Solving linear ODE with $\mathcal{L}$

Example: solve output $y(t)$

\[
\frac{d^2}{dt^2} y + 2 \frac{d}{dt} y + 3y = 4 \frac{d}{dt} u + 5u, \quad u(t), y(0), \dot{y}(0) \quad \text{given}
\]
Solving linear ODE with $\mathcal{L}$

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→ Laplace transform

\[
\text{LHS} = s^2 Y(s) - sy(0) - \dot{y}(0) + 2 \left( sY(s) - y(0) \right) + 3 Y(s)
\]
\[
= (s^2 + 2s + 3)Y(s) - (s + 2)y(0) - \dot{y}(0)
\]
\[
\text{RHS} = 4 \left( sU(s) - u(0) \right) + 5U(s) = (4s + 5)U(s) - 4u(0)
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Solving linear ODE with $\mathcal{L}$

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⇒ Set LHS = RHS and solve for $Y(s)$:

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Y(s) = \frac{4s + 5}{s^2 + 2s + 3} U(s) + \frac{s + 2}{s^2 + 2s + 3} y(0) + \frac{1}{s^2 + 2s + 3} (\dot{y}(0) - 4u(0))
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$\Rightarrow$ Compute $y(t) = \mathcal{L}^{-1}[y(t)]$ using $\mathcal{L}^{-1}$-transform (table)

Given: $U(s) = \mathcal{L}[u(t)]$, $u(0)$, $y(0)$ and $\dot{y}(0)$
Transfer function and impulse response
Transfer function $G(s)$

Assuming initial values are zero $y(0) = \dot{y}(0) = \cdots = 0$ and $u(0) = \dot{u}(0) = \cdots = 0$. Effect of input $u$ on output $y$:

$$Y(s) = \frac{G(s)}{s^2 + 2s + 3} U(s),$$

where $G(s)$ is the system transfer function $u \rightarrow y$. 

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where $G(s)$ is the system transfer function $u \rightarrow y$.

- More generally,

$$Y(s) = G(s)U(s)$$

is a model of the relation between the system input $u$ and output $y$. 
Transfer function

- A system described by the linear ODE

\[
\frac{d^n}{dt^n}y + \cdots + a_{n-1} \frac{d}{dt}y + a_n y = b_0 \frac{d^m}{dt^m}u + \cdots + b_{m-1} \frac{d}{dt}u + b_m u
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with initial values 0.

- Laplace transform of both sides:

\[
(s^n + \cdots + a_{n-1}s + a_n)Y(s) = (b_0 s^m + \cdots + b_{m-1}s + b_m)U(s)
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- Laplace transform of both sides:

\[ (s^n + \cdots + a_{n-1} s + a_n) Y(s) = (b_0 s^m + \cdots + b_{m-1} s + b_m) U(s) \]

- System transfer function is a *rational* function:

\[ G(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} \]

Note that \( s \) and \( G(s) \) are complex-valued!
Weighting function/impulse response

A system \( Y(s) = G(s)U(s) \) (at rest \( t = 0 \)) yields

\[ y(t) = \mathcal{L}^{-1} [Y(s)] = \int_0^t g(\tau)u(t - \tau)d\tau, \]

i.e. a convolution between \( u(t) \) and

\[ g(t) = \mathcal{L}^{-1} [G(s)] \]

denoted the system weighting function.
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Suppose input \( u(t) = \delta(t) = \text{(Dirac)pulse} \), then output

\[
y(t) = \int_0^t g(\tau)\delta(t - \tau)d\tau = g(t).
\]

Hence \( g(t) \) is called the system impulse response.
Poles, zeros and stability
Poles and zeros
Characterizing system behaviour

System with transfer function $G(s)$

- **Zeros:** $s'$ is a zero, if $G(s') = 0$.
- **Poles:** $s'$ is a pole, if $G(s')$ is a singularity, that is, $G'(s') = \pm \infty$. 
Poles and zeros
Characterizing system behaviour

\[ y(t) = G(s)u(t) \]

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- **Zeros:** \( s' \) is a zero, if \( G(s') = 0 \).

- **Poles:** \( s' \) is a pole, if \( G(s') \) is a singularity, that is, \( G(s') = \pm \infty \).

- If \( G(s) = \frac{B(s)}{A(s)} \) is a rational function
  
  - zeros = the roots to \( B(s) = 0 \),
  
  - poles = the roots to \( A(s) = 0 \).
Assume system $Y(s) = G(s)U(s)$, where $G(s) = \frac{B(s)}{A(s)}$. We want

$$y(t) = \int_{0}^{t} g(\tau)u(t - \tau)\,d\tau$$

where $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$. 

Denominator always factorize with roots/poles:

$A(s) = s^n + a_1s^{n-1} + \cdots + a_ns^n = (s + \sigma_1)(s + \sigma_2)\cdots(s + \sigma_j)^2 + \omega_j^2$ where poles are either

- real-valued: $-\sigma_1,\ldots,
- complex-conjugated: $-\sigma_j \pm i\omega_j,\ldots$
Poles and solution to linear ODE:s
Characterizing system behaviour

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▸ real-valued: \(-\sigma_1, \ldots\)

▸ complex-conjugated: \(-\sigma_j \pm i\omega_j, \ldots\)
Poles and solution to linear ODE:s
Characterizing system behaviour

▶ Now insert $A(s)$ and use partial-fraction decomposition

$$G(s) = \frac{B(s)}{A(s)} = \frac{\beta_1}{s + \sigma_1} + \cdots + \frac{B_j(s)}{(s + \sigma_j)^2 + \omega_j^2} + \cdots$$
Poles and solution to linear ODE:s
Characterizing system behaviour

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- Impulse response $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$ using table:

$$g(t) = \beta_1 e^{-\sigma_1 t} + \cdots + b_j e^{-\sigma_j t} \sin(\omega_j t + \varphi_j) + \cdots$$

and system output

$$y(t) = \int_0^t g(\tau) u(t - \tau) d\tau$$
Poles and solution to linear ODE:s
Characterizing system behaviour

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Output as linear combination of exponential functions
Output as linear combination of exponential functions

Real-parts of poles ($-\sigma$) play an important role
Stability
Characterizing system behaviour

Definition:
A system $Y(s) = G(s)U(s)$ is input-output stable if all bounded inputs $u(t)$ yield a bounded output $y(t)$.

Bounded signal $x(t)$ means $\Leftrightarrow |x(t)| \leq K$ for some $K$.

[Board: bounded impulse response + real-part of poles]
Stability
Characterizing system behaviour

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[Board: bounded impulse response + real-part of poles]

Result:
Assume \( G(s) = B(s)/A(s) \) with poles \( s = p_1, p_2, \ldots, p_n \) (and order of denominator \( \geq \) numerator)

\[ Y(s) = G(s)U(s) \text{ input-output stable } \Leftrightarrow \text{Re}\{p_i\} < 0 \]
Graphical representation of poles and zeros
Characterizing system behaviour

$$G(s) = \frac{B(s)}{A(s)} \text{ stable } \Leftrightarrow \text{ poles lie in left-halfplane}$$
Examples
Build intuition from simple systems

Ex. #1: Vehicle in motion

Standard form:

\[
\frac{d}{dt} y + \left( \frac{C}{m} \right) y = \left( \frac{1}{m} \right) u
\]

Figur: Force \( u(t) \) and velocity \( y(t) \).
Build intuition from simple systems

Ex. #2: Damper

Figur: Force $u(t)$ and position $y(t)$.

Standard form:

$$\frac{d^2}{dt^2} y + \left( \frac{K}{m} \right) y = \left( \frac{1}{m} \right) u$$

[Board: poles]
Build intuition from simple systems

Ex. #3: Inverted pendulum pendel

\[
\begin{align*}
\frac{d^2 y}{dt^2} - \left( \frac{3g}{2L} \right) y &= \left( \frac{3}{mL^2} \right) u \\
\text{Figur: Torque } u(t) \text{ and angle } y(t).
\end{align*}
\]

Standard form (around } y \approx 0):
Summary and recap

- Transfer functions as a system description
- Poles and zeros
- (Bounded) input-output stability