Time response, feedback and PID-control

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1) Systems with impulse response \( g(t) = \mathcal{L}^{-1}[G(s)] \), where \( G(s) \) is rational, are all
   a) Products of sinusoidal functions ↑
   b) Linear combinations of exponential functions ↑
   c) Stable ↓
Poles and time responses
Poles and time responses

Step response

\[ u(t) \rightarrow G \rightarrow y(t) \]

Model \( Y(s) = G(s)U(s) \) with transfer function

\[ G(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}. \]
Poles and time responses

**Step response**

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G(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}.
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Suppose input is a step:

\[
u(t) = \begin{cases} u_0 \text{ (const.)} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases}
\]

\[
\mathcal{L} \rightarrow U(s) = \frac{u_0}{s}
\]
Poles and time responses

Step response

\[ y(t) = G(s)U(s) \]

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\end{cases}
\]

\[
\mathcal{L} \left\{ u(t) \right\} \rightarrow U(s) = \frac{u_0}{s}
\]

Study step response \( y(t) \) of the system
Poles and time responses

Poles distance from the origin $\leftrightarrow$ quickness

Ex. 1st-order system:

$$G(s) = \left\lceil \text{typ. form} \right\rceil = \frac{p}{s + p}$$

Pole at:

$$s = -p$$

Pole-zero diagram:
Poles and time responses

Poles distance from the origin $\leftrightarrow$ quickness

Ex. 1st-order system:

$$G(s) = \left[ \text{typ. form} \right] = \frac{p}{s + p}$$

Step response:
Poles and time responses

Complex-conjugated poles ↔ system oscillations

Ex. 2nd-order system:

\[ G(s) = \left[ \text{alt. form} \right] = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \]

Poles at:

\[ s = -\omega_0\xi \pm i\omega_0 \sqrt{1 - \xi^2} \]

Pole-zero diagram:
Poles and time responses

Complex-conjugated poles $\leftrightarrow$ system oscillations

Ex. 2nd-order system:

$$G(s) = \left[ \text{alt. form} \right] = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

Step response:

![Graph showing step responses for different values of $\xi$]
Poles and time responses

Pole that is closest to the origin $\leftrightarrow$ slowest time constant

Ex.: 3rd-order system

$$G(s) = \frac{p}{s + p} \cdot \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

with $\omega_0 = 1$ and $\zeta = \sqrt{0.5}$.

Poles at:

$$s = -p \quad \text{och} \quad s = -\omega_0\xi \pm i\omega_0\sqrt{1 - \xi^2}$$

Pole-zero diagram:
Poles and time responses

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![Diagram](attachment:diagram.png)
Assume stable system \( Y(s) = G(s)U(s) \) with

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   u(t) = \begin{cases} 
   u_0 \text{ (const.)} & \text{for } t \geq 0, \\
   0 & \text{for } t < 0, 
   \end{cases}
\]

\[
   \mathcal{L} \rightarrow U(s) = \frac{u_0}{s}
\]

The system static gain is therefore \( G(0) \).
Static gain of a system

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    \end{cases}
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\[
    \mathcal{L} \quad U(s) = \frac{u_0}{s}
\]

Final value of step response \( y(t) \) can be computed using final value theorem:

\[
y_f = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s) \frac{u_0}{s} =
\]
Static gain of a system

Assume *stable* system $Y(s) = G(s)U(s)$ with

$$u(t) = \begin{cases} u_0 \text{ (const.)} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad \mathcal{L} \quad U(s) = \frac{u_0}{s}$$

*Final value* of step response $y(t)$ can be computed using *final value theorem*:

$$y_f = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s) \frac{u_0}{s} = G(0)u_0$$
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The system static gain is therefore $G(0)$
Connected and feedback systems
Connected and feedback systems

Transfer function obtained using Laplace + added help signals

Ex.: Parallel-connected systems

\[ Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)U(s) \]
\[ = (G_1(s) + G_2(s))U(s) \]
Connected and feedback systems

Transfer function obtained using Laplace + added help signals

Ex.: Serial-connected systems

\[ Y(s) = G_2(s)U_1(s) = G_2(s)(G_1(s)U(s)) = G_2(s)G_1(s)U(s) \]
Connected and feedback systems

Transfer function obtained using Laplace + added help signals

Ex.: Feedback systems

\[ u \rightarrow y \]

[Board: derive transfer function \( u \rightarrow y \)]
Feedback control based on error signal

PID-control

Feedback control:

Simple control strategy: Use the control error

\[ e(t) \triangleq r(t) - y(t) \]

[Board: intuition from temperature control]

to determine input \( u(t) \)
Feedback control based on error signal

PID-control

Feedback control:

Simple control strategy: Use the control error

\[ e(t) \triangleq r(t) - y(t) \]

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to determine input \( u(t) \) based on:

▶ current error: \( \propto e(t) \) (Proportional)
Feedback control based on error signal

PID-control

Feedback control:

Simple control strategy: Use the control error

\[ e(t) \triangleq r(t) - y(t) \]

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to determine input \( u(t) \) based on:

▶ current error: \( \propto e(t) \) (Proportional)
▶ past error: \( \propto \int_{\tau=0}^{t} e(\tau)d\tau \) (Integral)
Feedback control based on error signal

PID-control

Feedback control:

\[ e(t) \triangleq r(t) - y(t) \]

Simple control strategy: Use the control error to determine input \( u(t) \) based on:

- current error: \( \propto e(t) \) (Proportional)
- past error: \( \propto \int_{\tau=0}^{t} e(\tau)\,d\tau \) (Integral)
- change in error: \( \propto \dot{e}(t) \) (Derivative)
Ideal PID-controller

Controller with user parameters:

\[ u(t) = K_p e(t) + K_i \int_{\tau=0}^{t} e(\tau) d\tau + K_d \dot{e}(t) \]

Laplace domain:

\[ U(s) = K_p E(s) + K_i \frac{1}{s} E(s) + K_d s E(s) \]

\[ \text{controller } F(s) \]

\[ \text{system } G \]

\[ y \]
**Ideal PID-controller**

Controller with user parameters:

\[ u(t) = K_p e(t) + K_i \int_{\tau=0}^{t} e(\tau) d\tau + K_d \dot{e}(t) \]

Laplace domain:

\[ U(s) = (K_p + K_i s + K_d s) E(s) \]

Controller \( F(s) \) as \( E(s) \).
**Ideal PID-controller**

Controller with user parameters:

\[ u(t) = K_p e(t) + K_i \int_{\tau=0}^{t} e(\tau) d\tau + K_d \dot{e}(t) \]

- **P**roportional
- **I**ntegrating
- **D**erivative

Laplace domain:

\[ U(s) = K_p E(s) + K_i \frac{1}{s} E(s) + K_d s E(s) \]

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controller \( F(s) \)
Ideal PID-controller

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\[ = \left( K_p + \frac{K_i}{s} + K_d s \right) E(s). \]
Analysis of simple feedback control
Simple feedback control system

Closed-loop system from reference to output

- Closed-loop system from $r$ to $y$: $G_c(s)$
- Open-loop system from $e$ to $y$: $G_o(s) = G(s)F(s)$
Simple feedback control system
Closed-loop system from reference to output

Closed-loop system from \( r \) to \( y \): \( G_c(s) \)

Open-loop system from \( e \) to \( y \): \( G_o(s) = G(s)F(s) \)

[Board: derive closed-loop system \( G_c \)]

Note: We can design the poles of \( G_c \)!
Accuracy: stationary control error

Using a step as reference signal $r$

Assume stable $G_c(s)$ with reference (step):

$$r(t) = \begin{cases} r_0 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad \mathcal{L} \quad R(s) = \frac{r_0}{s}$$

*Final value of error $e(t)$*
Accuracy: stationary control error

Using a step as reference signal \( r \)

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 r(t) = \begin{cases} 
 r_0 & \text{for } t \geq 0, \\
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\end{cases} \quad \mathcal{L} \quad R(s) = \frac{r_0}{s} 
\]

**Final value of error** \( e(t) \)

\[
e_f = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + G_0(s)} \frac{r_0}{s} = \lim_{s \to 0} \frac{r_0}{1 + G(s)F(s)}
\]
Accuracy: stationary control error
Using a step as reference signal $r$

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$$r(t) = \begin{cases} r_0 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad \mathcal{L} \rightarrow \quad R(s) = \frac{r_0}{s}$$

**Final value of error** $e(t)$ \( \mathcal{L} \rightarrow \) $E(s) = R(s) - Y(s)$:

$$e_f = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + G_0(s)} \frac{r_0}{s}$$

$$= \lim_{s \to 0} \frac{r_0}{1 + G(s)F(s)}$$

**Result:**

Stationary error $e_f$ approaches 0 if $G(0)F(0) = \infty$. (E.g. when $F(s)$ contains $\frac{1}{s}$, i.e. integration.)
Summary and recap

- Relation between poles and system time-response
- The transfer function for connected and feedback systems
- Simple feedback control and ideal PID-controller
- Control accuracy: stationary error