Recapitulation
What do we want?

- The control problem: Minimize $e$, including the effect of $v$, and keep $u$ small ($u_{min} \leq u \leq u_{max}$).
- Ultimate course goal: Find the optimal solution!
- The controller is implemented in a computer ⇒ must be described as a discrete-time filter.
Recapitulation, cont’d
What have we achieved? — summary of what we have looked at so far

- **Discrete-time systems:**
  - Difference equations,
  - shift operator $q$,
  - stability region = inside of the unit circle.

- **Sampling of systems = zero-order hold sampling (ZOH):**
  - Exact for $t = kh$,
  - sampling period = $h$,
  - Nyquist frequency $\omega_n = \omega_s/2 = \pi/h$,
  - frequency response of $G(z)$ for $z = e^{i\omega h}$.

- **MIMO systems:** Straightforward to use state space forms.

- **Disturbance models:**
  - Spectral density: $\Phi(\omega) = \mathcal{F}[r(\tau)]$,
  - white noise $v \leftrightarrow \Phi_v(\omega) = R_v = \text{const.}$,
  - linear filtering: $y = Gu \Rightarrow \Phi_y = |G|^2 \Phi_u$,
  - spectral factorization: $\Phi = |G|^2 R$,
  - Lyapunov equation $\Rightarrow \Pi_x = Exx^T$.

- **Kalman filters:** Optimal observer, Riccati eq. (CARE, DARE).
Control design
Starting point in the continuous-time case

- Use the "standard" state space representation

\[
\begin{align*}
\dot{x} &= Ax + Bu + Nv_1, \\
z &= Mx, \\
y &= Cx + v_2,
\end{align*}
\]

\[
\eta = \begin{bmatrix} v_1 \\
v_2 \end{bmatrix}, \quad \Phi_\eta(\omega) = \begin{bmatrix} R_1 & R_{12} \\
R_{12}^T & R_2 \end{bmatrix}
\]

- Minimize the criterion

\[
V = \|z\|_Q^2 + \|u\|_Q^2 = E \left[ z^T Q_1 z + u^T Q_2 u \right]
\]

- The weighting matrices,

\[
Q_1 = Q_1^T \geq 0 \quad \text{and} \quad Q_2 = Q_2^T > 0,
\]

are design parameters.
Control strategy
State feedback with observer

The optimal controller is conveniently represented as state feedback from estimated states.

- **Control law:** \( u(t) = -L\hat{x}(t) + \hat{r}(t) \)
- **Observer:** \( \dot{\hat{x}}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t)) \)
- The control law can also be written as

\[
U(s) = F_r(s)\tilde{R}(s) - F_y(s)Y(s),
\]
\[
F_y(s) = L(sI - A + BL + KC)^{-1}K,
\]
\[
F_r(s) = 1 - L(sI - A + BL + KC)^{-1}B
\]

- The poles of the closed loop system are the roots of

\[
0 = \det(sI - A + BL) \cdot \det(sI - A + KC),
\]

i.e. poles from state feedback + the observer poles.
LQG: The optimal controller

Theorem 9.1

- The optimal control law is $u(t) = -L\hat{x}(t)$,
- $\hat{x}(t)$ is obtained from the corresponding Kalman filter.
- The optimal state feedback gain is
  $$L = Q_2^{-1}B^T S,$$
  where $S = S^T \geq 0$ is the solution to the continuous-time Riccati equation (CARE)
  $$0 = A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S$$
  - N.B. There are two different CAREs involved, the one above and the one for the Kalman filter!
- Some technical conditions: $(A, B)$ stabilizable, $(A, C)$ and $(A, M^T Q_1 M)$ detectable...
LQ/LQG: Properties

- $A - BL$ is always stable
- The control law $u = -Lx(t)$ (pure state feedback) is optimal, also for the deterministic case ($v_1 = 0$ and $v_2 = 0$) $\Leftrightarrow$ LQ = linear quadratic control.
- If $v_1$ and $v_2$ have Gaussian distributions the controller is the optimal controller (Cor. 9.1) $\Leftrightarrow$ LQG = linear quadratic Gaussian control.
- Theorem 9.1 = the separation theorem: The optimal observer = the Kalman filter, combined with the optimal state feedback (LQ) give the optimal controller! (This is far from obvious...)
- The LQ/LQG controller looks exactly the same for SISO and MIMO systems.
The servo problem
How can the reference signal $r(t)$ be included?

- **General solution:** Characterize $r(t)$ by its spectrum and model it in the same way as a disturbance, i.e. incorporate it in the model.

- **Special case, Theorem 9.2:** If $r(t)$ is piecewise constant, then the criterion

$$V = \|z - r\|_{Q_1}^2 + \|u - u^*(r)\|_{Q_2}^2$$

is minimized with the control law

$$u(t) = -L\hat{x}(t) + L_r r(t),$$

where $L$ and $\hat{x}(t)$ are given as in Theorem 9.1, and $L_r$ is chosen so that

$$I = M(sI - A + BL)^{-1}BL_r\big|_{s=0} = G_c(0)L_r.$$

(Then $u^*(r) = L_r r$.)
Example: LQ control of a DC-motor

The effect of $Q_1$ and $Q_2$

- The DC-motor $Y(s) = \frac{1}{s(s+1)}U(s)$.
- Design parameters $Q_1 = 1 = \text{constant}$ and $Q_2$ varied.
- LQ control $\Rightarrow$ pure state feedback: $u(t) = -Lx(t) + mr(t)$.

Simulations: Step responses ($r = \text{unit step}$) for the closed loop systems. The output $y$ to the left, the input $u$ to the right.